Lecture 17: Representations of symmetric groups, II

1) Completing the classification of irreducibles.

2) Character formula.

Ref: [E], Sec. 5.12, 5.13, 5.15.

1) Completing the classification of irreducibles.

1.0) Reminder

We consider representations of $S_n$ over an algebraically closed field $F$ of char 0. We want to prove that the irreducibles are labelled by the partitions of $n$ (a.k.a. Young diagrams w. $n$ boxes), $\lambda \leftrightarrow V_\lambda$. Here $V_\lambda$ is the unique irreducible occurring in both $I_\lambda^+ = \text{Ind}_{S_\lambda}^{S_n} \text{triv}$ & $I_\lambda^- = \text{Ind}_{S_\lambda^{op}}^{S_n} \text{sgn}$.

**Example:** $I_{(n)}^+ = \text{Ind}_{S_n}^{S_n} \text{triv} = \text{triv} \Rightarrow V_{(n)} = \text{triv}$. Further, $I_{(n-1,1)}^+ = \text{Ind}_{S_{n-1}}^{S_n} \text{triv} = \text{Fun}(S_n/S_{n-1}(\sim \{1,2,\ldots,n\}), F) = F^n$, permutation representation. We know, Lec 5, that $F^n = \text{triv} \oplus F^n$, the decomposition into irreducibles. Since $\lambda \mapsto V_\lambda$ is a bijection & $V_{(n)} = \text{triv}$, we see that $V_{(n-1,1)} = F^n$. 

In Lec 16 we reduced the proof of the bijection to

**Main Claim:** For partitions \( \lambda, \mu \) of \( n \), we have:

1) \( \text{Hom}_{S_n}(I^\lambda_\lambda, I^-_\mu) \neq 0 \Rightarrow \mu^t \leq \lambda^t \)

2) \( \dim \text{Hom}_{S_n}(I^+_\lambda, I^-_\lambda) = 1 \)

We have proved a lemma that we'll use for Main Claim:

**Lemma:** \( \dim \text{Hom}_{S_n}(I^+_\lambda, I^-_\mu) \) coincides with
\[
\dim \{ f : S_n \rightarrow \mathbb{F} | f(\tau g \delta) = \text{sgn}(\tau) f(g) \neq g \in S_n, \tau \in S_{\mu^t}, g \in S_{\lambda^t} \} \tag{1}
\]

1.1) **Combinatorial preparation**

We start with a combinatorial formula for (1). We need some notation. Let \( k \) be the number of parts in \( \lambda \) and consider the subsets \( X_i, i = 1, \ldots, k \), of \( \{1, \ldots, n\} \) given by
\[
X_i = \{ \lambda_{i+1} + \ldots + \lambda_{i+m} | m = 1, \ldots, \lambda_i, \lambda_i \} \quad \text{so that} \quad S_\lambda = \{ \delta \in S_n | \delta(X_i) = X_i \cup \{ \lambda_i \} \}
\]
Similarly, let \( l \) be the number of parts in \( \mu^t \) and consider the subsets \( Y_j, j = 1, \ldots, l \),\n\[
Y_j = \{ \mu_{j+1}^t + \ldots + \mu_{j+m}^t | m = 1, \ldots, \mu_j^t, \mu_j^t \} \quad \text{so that} \quad S_{\mu^t}
\]
\[ S_{\mu^t} = \{ \tau \in S_n \mid \tau(y_j) = y_j \ \forall j \} \]

**Lemma:** (1) coincides w the number of double cosets
\[ S_{\mu^t} g S_\lambda \in S_{\mu^t} \backslash S_n / S_\lambda \text{ s.t.} \]
\[ (\ast) \ |g^{-1} j^i X_i j| \leq 1 \ \forall i = 1, \ldots, k, j = 1, \ldots, l \]

Note that whether or not (*) holds for a given double coset is independent of the choice of \( g \) in that coset *(exercise).*

**Proof of Lemma:** Suppose (*) fails. We claim that:
\[ f(\tau g g') = \text{sgn}(\tau) f(g) \ \forall \tau \in S_{\mu^t}, g' \in S_\lambda \Rightarrow f(g) = 0. \] Indeed, let \( a, b \in g^{-1} j^i X_i, a \neq b. \) Then \( g' = (a, b) \) preserves all \( X_i \), hence \( g' \in S_\lambda. \) Similarly \( g g' \in S_n \Rightarrow \tau = (g g') \in S_{\mu^t}. \) But
\[ \tau = g' g^{-1} \Rightarrow \tau g = g' g^{-1} \Rightarrow \]
\[ -f(g) = \text{sgn}(\tau) f(g) = f(\tau g) = f(g' g^{-1}) = f(g) \Rightarrow f(g) = 0. \]

Now let \( S_{\mu^t} g_1 S_\lambda, \ldots, S_{\mu^t} g_r S_\lambda \) be all double cosets s.t. (*) holds. It is an exercise to show that for \( s = 1, \ldots, r: \]
Define \( f_5 : S_n \to \mathbb{F} \), by requiring that \( f_5 \) is zero on all double cosets but \( S_{\mu} g S_{\chi} \) & \( f_5(\tau g_6') := \text{sgn}(\tau) \), which is well-defined by of the uniqueness condition above, so that \( f_5(\tau g_6') = \text{sgn}(\tau) f_5(g) \). The following finishes the proof.

**Exercise:** The functions \( f_5, ..., f_1 \) form a basis in the space in (1) (hint: \( f = \sum_{i=1}^{r} f(g_i) f_i \)) hence \( (1) = r \). \( \square \)

1.2) Proof of Main Claim

Thanks to lemmas in Sections 1.0, 1.1, we need to show that the existence of a double coset satisfying (\( * \)) implies \( \mu^t \leq \lambda^t \) (which we'll prove) & for \( \lambda = \mu \), there's the unique such double coset (which we'll see in the course of the proof). 

\( \lambda_q^t \) is the height of the \( q \)-th column in \( \lambda \) viewed as a diagram. So, \( \kappa \) (the number of \( X_i \)'s) = \( \lambda_q^t \) & more generally:
(2) \( \lambda_t^q = \{ i \mid |X_i| \geq q \} \neq \emptyset \), \( q = 1, \ldots, \lambda_t \) (exercise)

If \( \mu_t^q = |Y_q| > \lambda_t \), then (*) is violated. We only need to analyze the case \( \mu_t^q = \lambda_t \) (otherwise, \( \mu^t < \lambda^t \) indeed). In this case, (*) \( \Rightarrow |g^{-1}_i Y_q \cap X_i| = 1 \neq \lambda_t \). Let \( b_i \in S_{\lambda_i} \), \( b_i = b_{i_1} \ldots b_{i_k} \), where \( b_{i_k} \) is the transposition in \( S_{\lambda_i} = \text{Bij} (X_i) \) permuting the only element \( g_i^{-1} Y_q \cap X_i \) with the smallest element in \( X_i \), \( \lambda_i + \lambda_{i-1} + 1 \). Replacing \( g_i \) with \( b_i \), we get \( g_i^{-1} Y_q \cap X_i = \{ \lambda_i + \lambda_{i-1} + 1 \} \).

Set \( X_i' = X_i \setminus \{ \lambda_i + \lambda_{i-1} + 1 \} \)

- \( g_i^{-1} Y_q \cap X_i' = g_i^{-1} Y_q \cap X_i' \neq j \geq 2, \forall i \).
- \( \# \{ i \mid X_i' \neq \emptyset \} = \lambda_t^t, \) by (2).

So, if \( \mu_t^q = |Y_q| > \lambda_t \), then \( g_i^{-1} Y_q \cap X_i' \) contains 2 elements for some \( i \), violating (*). So we can assume \( \mu_t^q = \lambda_t \), modify \( g \) similarly to the above, form \( X_i'' = X_i' \setminus \{ \lambda_i + \lambda_{i-1} + 2 \} \) & continue in the same fashion proving: \( \exists \) coset satisfying (*) \( \Rightarrow \mu^t \leq \lambda^t \).

If \( \lambda = \mu \), then we have modified \( g \) by multiplying \( \lambda \) with an element of \( S_{\lambda} \) from the right so that

(3) \( g_i^{-1} Y_q \cap X_i = \{ \lambda_i + \lambda_{i-1} + j \} \) if \( j \leq \lambda_i \) (and \( \emptyset \) else).
An element \( g \in S_n \) satisfying (3) (and hence (\( \ast \)) exists: to see this: fill the diagram \( \lambda \) w. numbers 1, 2, ..., \( n \), left to right, then bottom to top. e.g. for \( \lambda = (3, 1) \) we get

\[
\begin{array}{c}
4 & 5 \\
2 & 3 \\
1 & \end{array}
\]

The \( i \)th row consists of the elements of \( X_i \).

Then fill the same diagram but now bottom to top & then left to right.

\[
\begin{array}{c}
2 & 4 \\
1 & 3 & 5 \\
\end{array}
\]

The \( j \)th column consists of the elements of \( Y_j \).

Take \( g \) that sends the number in the first diagram to the number in the same box in the 2nd diagram. In our example it is \( 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 5, 4 \mapsto 2, 5 \mapsto 4 \). It satisfies (3).

We claim that (3) determines \( g \) uniquely in its left \( S_{\mu'} \)-coset finishing the proof. Indeed, let \( g_1 \) be another element satisfying (3). Let \( \tau \in S_{\mu'} \) be defined by sending

\[
\tau(\lambda_1 + \ldots + \lambda_i + j) \in Y_j \ to \ g_1(\lambda_1 + \ldots + \lambda_i + j) \in Y_j \ (i \leq \lambda_i). \]

This
uniquely determines the element \( \tau \) and it preserves each \( Y_j \), so \( \tau \) lies in \( S_n \). By construction, \( \tau g = g_1 \).

Rem: Note that the proof works as long as \( \text{char } F \neq 2 \).

2) Character formulas.

We proceed to computing characters of irreducible representations of symmetric groups. We assume that \( F \) is algebraically closed of characteristic 0 (the condition of being algebraically closed can be removed).

For \( m \in \mathbb{Z} \), define the power symmetric polynomial

\[
p_m = \sum_{i=1}^{N} x_i^m
\]

For \( \sigma \in S_n \), let \((m_1, \ldots, m_k)\) be its cycle type. We set

\[
p_{\sigma} = p_{m_1} p_{m_2} \cdots p_{m_k}
\]

For example, \( p_{\sigma} = p_{(1, \ldots, 1)} = (x_1 + \cdots + x_N)^n \).

Note that \( p_{\sigma} \) only depends on the conjugacy class of \( \sigma \).

Finally, we will need the Vandermonde determinant

\[
\Delta = \prod_{1 \leq i < j \leq N} (x_i - x_j) = \det (x_i^{N-j})_{i,j=1}^N
\]
**Theorem (Frobenius)** $X^\lambda(\nu)$ coincides with the coefficient of $\prod_{i=1}^N x_i^{\lambda_i + N - i}$ in $\Delta p_\nu$ (recall that we adjoin 0's to $\lambda$ if needed).

We will prove the theorem next time.

**Example:** Suppose $\lambda = (n)$, so that $I^\lambda = \text{Ind}_{S_n}^{S_n} \text{triv} = \text{triv}$

$\Rightarrow V^\lambda = \text{triv} \Rightarrow X^\lambda(\nu) = 1 \neq \nu \in S_n$. The monomial we care about is $x_1^{N-1+n} x_2^{N-2} \cdots x_{N-1}$. On the other hand,

$\Delta p_\nu = \left[\text{use the determinant description of } \Delta \right]$ 

$= \left( \sum_{\tau \in S_n} \text{sgn}(\tau) x_1^{\tau(1)-1} \cdots x_N^{\tau(n)-1} \right) p_\nu$. Note that the largest monomial (w.r.t. lexicographic order) in the sum is $x_1^{N-1} x_2^{N-2} \cdots x_{N-1}$, w. coefficient 1

while in $p_\nu$, it's $x_n^n$ w. coefficient 1. So $x_1^{N-1+n} x_2^{N-2} \cdots x_{N-1}$ is the largest monomial in $\Delta p_\nu$, & the coefficient is indeed 1.

**Remarks:** 1) The theorem is not easy to use to compute the characters (try to derive the formula for the character of $V^{(n+1, n)}$). It does, however, lead to fairly explicit combinatorial
torial results of which I would like to mention two:

- the formula for decomposition of $\text{Res}_{S_n,1} V_\lambda$ into the direct sum of irreducibles, Homework 4.

- the hook-length formula for $\dim V_\lambda = \text{coefficient of } \prod_{i=1}^N x_i^{\lambda_i + N - i}$ in $\Delta(x_1^{\lambda_1}, \ldots, x_N^{\lambda_N})$, see Sec 5.17 in [E].

We'll have a further discussion of how to think about the Frobenius character formula in a bonus lecture.

2) As was discussed in the bonus section of Lec 16, the results of this section carry over to the case of nonclosed char 0 fields. They also carry over to the case when $n \nmid p$.

But, in general, the characters (and even dimensions) of irreducible representations of $\text{Res}_{S_n} (w. \text{char } \mathbb{F} = p)$ are not known, and this is an area of active current research.