Lecture 17: Representations of symmetric groups, II

1) Completing the classification of irreducibles.
2) Character formula.

Ref: [E], Sec. 5.12, 5.13, 5.15.

1) Completing the classification of irreducibles.

1.0) Reminder

We consider representations of S_n over an algebraically closed field $\mathbb{F}$ of char 0. We want to prove that the irreducibles are labelled by the partitions of n (a.k.a. Young diagrams w. n boxes), $\lambda \xleftrightarrow{\sim} V_\lambda$. Here V_λ is the unique irreducible occurring in both $I^+_\lambda = \operatorname{Ind}_{S_\lambda}^{S_n} triv$ & $I^-_\lambda = \operatorname{Ind}_{S_{\lambda^t}}^{S_n} sgn$.

Example: $I^+_{(n)} = \operatorname{Ind}_{S_n}^{S_n} triv = triv \Rightarrow V_{(n)} = triv$. Further, $I^+_{(n-1,1)} = \operatorname{Ind}_{S_{n-1}}^{S_n} triv = \operatorname{Fun}(S_n/S_{n-1} (\simeq \{1,2,..n\}), \mathbb{F}) = \mathbb{F}^n$, permutation representation). We know, Lec 5, that $\mathbb{F}^n = triv \oplus \mathbb{F}^n_0$, the decomposition into irreducibles. Since $\lambda \mapsto V_\lambda$ is a bijection & $V_{(n)} = triv$, we see that $V_{(n-1,1)} = \mathbb{F}^n_0$.

In Lec 16 we reduced the proof of the bijection to

Main Claim: For partitions λ, μ of n, we have:

1) $\mathrm{Hom}_{S_n}(I^+_\lambda, I^-_\mu) \neq 0 \Rightarrow \mu^t \leq \lambda^t$.
2) $\dim \mathrm{Hom}_{S_n}(I^+_\lambda, I^-_\lambda) = 1$.

We have proved a lemma that we'll use for Main Claim:

Lemma: $\dim \mathrm{Hom}_{S_n}(I^+_\lambda, I^-_\mu)$ coincides w.

$$\dim\{f: S_n \to \mathbb{F} \mid f(\tau g \sigma) = \mathrm{sgn}(\tau) f(g) \ \forall g \in S_n, \tau \in S_{\mu^t}, \sigma \in S_\lambda\} \quad (1)$$

1.1) Combinatorial preparation.

We start w. a combinatorial formula for (1). We need some notation. Let k be the number of parts in λ & consider the subsets $X_i, i=1,\dots k$, of $\{1,2,\dots n\}$ given by $X_i = \{\lambda_1 + \dots + \lambda_{i-1} + m \mid m = 1,2,\dots,\lambda_i\}$ so that $S_\lambda = \{\sigma \in S_n \mid \sigma(X_i) = X_i \ \forall i\}$. Similarly, let ℓ be the number of parts in μ^t & consider the subsets $Y_j, j=1,\dots \ell$, $Y_j = \{\mu^t_1 + \dots + \mu^t_{j-1} + m \mid m = 1,2,\dots,\mu^t_j\}$ so that

$S_{\mu^t} = \{\tau \in S_n \mid \tau(Y_j) = Y_j \ \forall j\}.$

Lemma: (1) coincides w. the number of double cosets
$S_{\mu^t}\, g\, S_\lambda \in S_{\mu^t} \backslash S_n / S_\lambda$ s.t.
$(*)$ $|g^{-1}Y_j \cap X_i| \leq 1 \ \forall i = 1, \ldots k, \ j = 1, \ldots, \ell$

Note that whether or not $(*)$ holds for a given double coset is independent of the choice of g in that coset (exercise).

Proof of Lemma: Suppose $(*)$ fails. We claim that:
$f(\tau g \sigma) = \operatorname{sgn}(\tau) f(g) \ \forall \tau \in S_{\mu^t}, \sigma \in S_\lambda \Rightarrow f(g) = 0$. Indeed, let $a, b \in g^{-1}Y_j \cap X_i, a \neq b$. Then $\sigma = (a, b)$ preserves all $X_?$, hence $\sigma \in S_\lambda$. Similarly $ga, gb \in Y_j \Rightarrow \tau = (ga, gb) \in S_{\mu^t}$. But
$\tau = g\sigma g^{-1} \Rightarrow \tau g = g\sigma \Rightarrow$
$-f(g) = \operatorname{sgn}(\tau) f(g) = f(\tau g) = f(g\sigma) = f(g) \Rightarrow f(g) = 0.$

Now let $S_{\mu^t} g_1 S_\lambda, \ldots, S_{\mu^t} g_r S_\lambda$ be all double cosets s.t. $(*)$ holds. It is an exercise to show that for $s = 1, \ldots, r$:

$(*) \Leftrightarrow g_s^{-1} S_{\mu^t} g_s \cap S_\lambda = \{e\} \Leftrightarrow$

$\forall g \in S_{\mu^t} g_s S_\lambda \ \exists! \ \tau \in S_{\mu^t}, \sigma \in S_\lambda \mid g = \tau g_s \sigma.$

Define $f_s : S_n \to \mathbb{F}$, by requiring that f_s is zero on all double cosets but $S_{\mu^t} g_s S_\lambda$ & $f_s(\tau g_s \sigma) := \mathrm{sgn}(\tau)$, which is well-defined b/c of the uniqueness condition above, so that $f_s(\tau g \sigma) = \mathrm{sgn}(\tau) f(g)$. The following finishes the proof.

Exercise: The functions $f_1, \dots, f_r$ form a basis in the space in (1) (hint: $f = \sum_{i=1}^{r} f(g_i) f_i$) hence $(1) = r$. $\square$

1.2) Proof of Main Claim

Thanks to lemmas in Sections 1.0, 1.1, we need to show that the existence of a double coset satisfying $(*)$ implies $\mu^t \leq \lambda^t$ (which we'll prove) & for $\lambda = \mu$, there's the unique such double coset (which we'll see in the course of the proof).

λ_q^t is the height of the q-th column in λ viewed as a diagram. So, k (the number of X_i's) $= \lambda_1^t$ & more generally:

(2) $\lambda^t_q = |\{i \mid |X_i| \geq q\}| \ \forall\, q = 1, \dots \lambda_1$ (exercise)

If $\mu_1^t = |Y_1| > k$, then (*) is violated. We only need to analyze the case $\mu_1^t = \lambda_1^t$ (otherwise, $\mu^t < \lambda^t$ indeed). In this case, (*) $\Rightarrow |g^{-1}Y_1 \cap X_i| = 1 \ \forall\, i = 1, \dots, k$. Let $\sigma \in S_\lambda$, $\sigma = \sigma_1 \dots \sigma_k$, where σ_i is the transposition in $S_{\lambda_i} = \mathrm{Bij}(X_i)$ permuting the only element in $g^{-1}Y_1 \cap X_i$ w. the smallest element in in X_i, $\lambda_1 + \dots + \lambda_{i-1} + 1$. Replacing g w. $g\sigma$, we get $g^{-1}Y_1 \cap X_i = \{\lambda_1 + \dots + \lambda_{i-1} + 1\}$

Set $X_i' := X_i \setminus \{\lambda_1 + \dots + \lambda_{i-1} + 1\} \Rightarrow$

- $g^{-1}Y_j \cap X_i = g^{-1}Y_j \cap X_i' \ \forall\, j \geq 2, \forall\, i$.
- & $\#\{i \mid X_i' \neq \emptyset\} = \lambda_2^t$, by (2).

So, if $\mu_2^t = |Y_2| > \lambda_2^t$, then $g^{-1}Y_2 \cap X_i'$ contains 2 elements for some i, violating (*). So we can assume $\mu_2^t = \lambda_2^t$, modify g similarly to the above, form $X_i'' = X_i' \setminus \{\lambda_1 + \dots + \lambda_{i-1} + 2\}$ & continue in the same fashion proving: $\exists$ coset satisfying (*) $\Rightarrow \mu^t \leq \lambda^t$.

If $\lambda = \mu$, then we have modified g by multiplying w. an element of S_λ from the right so that

(3) $g^{-1}Y_j \cap X_i = \{\lambda_1 + \dots + \lambda_{i-1} + j\}$ if $j \leq \lambda_i$ (and $\emptyset$ else).

An element $g \in S_n$ satisfying (3) (and hence ($*$)) exists: to see this: fill the diagram λ w. numbers $1, 2, \dots n$, left to right, then bottom to top: e.g for $\lambda = (3,2)$ we get

$\begin{array}{|c|c|c}\hline 4 & 5 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$ The ith row consists of the elements of X_i

Then fill the same diagram but now bottom to top & then left to right.

$\begin{array}{|c|c|c}\hline 2 & 4 & \\ \hline 1 & 3 & \multicolumn{1}{c|}{5} \\ \hline\end{array}$ The jth column consists of the elements of Y_j.

Take g that sends the number in the first diagram to the number in the same box in the 2nd diagram. In our example it is $1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 5$, $4 \mapsto 2$, $5 \mapsto 4$. It satisfies (3).

We claim that (3) determines g uniquely in its left S_{μ^t}-coset finishing the proof. Indeed, let g_1 be another element satisfying (3). Let $\tau \in S_{\mu^t}$ be defined by sending $g(\lambda_1 + \dots + \lambda_{i-1} + j) \in Y_j$ to $g_1(\lambda_1 + \dots + \lambda_{i-1} + j) \in Y_j$ (if $j \leq \lambda_i$). This uniquely determines the element τ and it preserves each Y_j, so lies in S_{μ^t}. By construction, $\tau g = g_1$. $\square$

Rem: Note that the proof works as long as char $\mathbb{F} \neq 2$.

2) Character formulas.

We proceed to computing characters of irreducible representations of symmetric groups. We assume that $\mathbb{F}$ is algebraically closed of characteristic 0 (the condition of being algebraically closed can be removed).

For $m \in \mathbb{Z}$, define the power symmetric polynomial

$$p_m := \sum_{i=1}^{N} x_i^m$$

For $\sigma \in S_n$, let $(m_1, \ldots m_k)$ be its cycle type. We set

$$p_\sigma := p_{m_1} p_{m_2} \cdots p_{m_k}$$

For example, $p_e = p_{(1,\ldots,1)} = (x_1 + \ldots + x_N)^n$.

Note that p_σ only depends on the conjugacy class of σ.

Finally, we will need the Vandermonde determinant

$$\Delta := \prod_{1 \leq i < j \leq N} (x_i - x_j) = \det\left(x_i^{N-j}\right)_{i,j=1}^{N}.$$

Theorem (Frobenius) $\chi_{V_\lambda}(\sigma)$ coincides w. the coefficient of $\prod_{i=1}^{N} x_i^{\lambda_i+N-i}$ in Δp_σ (recall that we adjoin 0's to λ if needed).

We will prove the theorem next time.

Example: Suppose $\lambda=(n)$, so that $I_\lambda = \mathrm{Ind}_{S_n}^{S_n} \mathrm{triv} = \mathrm{triv}$ $\Rightarrow V_\lambda = \mathrm{triv} \Rightarrow \chi_{V_\lambda}(\sigma)=1\ \forall \sigma \in S_n$. The monomial we care about is $x_1^{N-1+n} x_2^{N-2} \dots x_{N-1}$. On the other hand,

$$\Delta p_\sigma = [\text{use the determinant description of } \Delta]$$
$$=\Big(\sum_{\tau\in S_n} \mathrm{sgn}(\tau)\, x_1^{\tau(1)-1} \cdots x_N^{\tau(N)-1}\Big) p_\sigma.$$ Note that the largest monomial (w.r.t. lexicographic order) in the sum is

$$x_1^{N-1} x_2^{N-2} \dots x_{N-1} \text{ w. coefficient } 1$$

while in p_σ it's x_1^n w. coefficient 1. So $x_1^{N-1+n} x_2^{N-2} \dots x_{N-1}$ is the largest monomial in Δp_σ & the coefficient is indeed 1.

Remarks: 1) The theorem is not easy to use to compute the characters (try to derive the formula for the character of $V_{(n-1,1)}$). It does, however, lead to fairly explicit combinatorial results of which I would like to mention two:

- the formula for decomposition of $\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda$ into the direct sum of irreducibles, Homework 4.

- the hook-length formula for $\dim V_\lambda$ = coefficient of $\prod_{i=1}^{N} x_i^{\lambda_i+N-i}$ in $\Delta(x_1+\ldots+x_N)^n$; see Sec 5.17 in [E].

We'll have a further discussion of how to think about the Frobenius character formula in a bonus lecture.

2) As was discussed in the bonus section of Lec 16, the results of this section carry over to the case of nonclosed char 0 fields. They also carry over to the case when $n > p$. But, in general, the characters (and even dimensions) of irreducible representations of $\mathbb{F}S_n$ (w. char $\mathbb{F}=p$) are not known, and this is an area of active current research.