Lecture 19: Finite dimensional associative algebras, I.

1) Introduction.
2) Basics on modules.
3) Bonus: an infinite dimensional simple algebra

1) Introduction.

We now switch our focus and instead of representations of finite groups look at those of finite dimensional associative algebras (over a field $\mathbb{F}$). Of course, the two are connected: a representation of G is the same thing as a representation of its group algebra $\mathbb{F}G$. And, in fact, we will (re)prove some claims about representations of finite groups in this section.

We will mainly focus on three classes of algebras simple algebras, semisimple algebras, & skew-fields. The definition of a skew-field was given in Sec 3 of Lec 7: these are associative rings s.t. every nonzero element is invertible. The other two classes are defined below.

1.1) Simple rings (and algebras)

Let's proceed to simple rings (& algebras). Let A be an associative (unital) ring.

Definition 1: A two-sided ideal in A is a subgroup I (w.r.t. $+$) s.t. $a \in A, b \in I \Rightarrow ab, ba \in I$.

Here's a reason for this definition: if $\varphi: A \to B$ is a homomorphism of associative rings, then $\ker \varphi$ is a two-sided ideal. Conversely, every two-sided ideal arises as such a kernel: A/I has the unique associative ring structure s.t. the projection map $\pi: A \longrightarrow A/I$, $a \mapsto a+I$, is a ring homomorphism. Of course, $I = \ker \pi$.

Definition 2: An associative ring A is simple if it has exactly 2 two-sided ideals: $\{0\}$ & A. An associative $\mathbb{F}$-algebra is called simple if it is simple as a ring.

Example: Let S be a skew-field, and $n \in \mathbb{Z}_{>0}$. Then we can form the ring $\mathrm{Mat}_n(S)$ of $n \times n$-matrices w. entries in S. We claim it's simple. As usual, let E_{ij} denote the matrix unit at position (i,j). Then, for $B = (b_{ij}) \in \mathrm{Mat}_n(S)$, we have

(1) $$E_{ii} B E_{jj} = b_{ij} E_{ij}$$

Now let I be a nonzero two-sided ideal in $\mathrm{Mat}_n(S)$ and $B \in I$, $B \neq 0$. If $b_{ij} \neq 0$ for some i,j, then b_{ij} is invertible & thx to (1) $\Rightarrow E_{ij} \in I \Rightarrow E_{kj} = E_{ki} E_{ij}$ & $E_{k\ell} = E_{kj} E_{j\ell} \in I$ $\forall\, k, \ell \Rightarrow I = \mathrm{Mat}_n(S)$.

The 1st important result in this part is a characterization of <u>finite dimensional</u> simple associative $\mathbb{F}$-algebras.

Theorem 1: Every finite dimensional simple assoc've $\mathbb{F}$-algebra is isomorphic to $\mathrm{Mat}_n(S)$ for the unique $n > 0$ & finite dimensional skew-field S over $\mathbb{F}$.

Remarks: 1) When $\mathbb{F}$ is algebraically closed, the only such S is $\mathbb{F}$ itself. Indeed, for $s \in S$ $\exists$ $m > 0$ s.t. $s^m = a_{m-1}s^{m-1} + \ldots + a_0$ for some $a_{m-1}, \ldots, a_0 \in \mathbb{F}$, here we use $\dim_{\mathbb{F}} S < \infty$. If $\alpha_1, \ldots \alpha_m \in \mathbb{F}$ are roots of the polynomial $x^m - a_{m-1}x^{m-1} - \ldots - a_0$, then $s^m - a_{m-1}s^{m-1} - \ldots - a_0 = \prod_{i=1}^{m}(s - \alpha_i)$ in S. Since S is a skew-field, we get $s = \alpha_i$ for some $i \Rightarrow s \in \mathbb{F} (\subset S) \Rightarrow S = \mathbb{F}$.

2) Outside of the setting of finite dimensional algebras, classifying simple rings is very hard.

1.2) Finite dimensional semisimple algebras.

Let A be a finite dimensional algebra over $\mathbb{F}$.

Definition: We say that A is *semisimple* if all its finite dimensional modules are completely reducible.

The following example also serves as a motivation.

Example: Let G be a finite group s.t. char $\mathbb{F}$ doesn't divide $|G|$. Then $\mathbb{F}G$ is semisimple (Maschke's Thm, Sec 2 of Lec 5). So finite dimensional semisimple algebras generalize group algebras $\mathbb{F}G$ under the assumptions of the example.

The names "simple" and "semisimple" look similar but the definitions do not. In fact, these two classes are very closely related. To state this relation note that the direct sum $A_1 \oplus \ldots \oplus A_k$ of associative algebras $A_i, i=1,\ldots,k$, carries a natural associative algebra structure (w. component-wise multiplication).

The following theorem will be proved later in the course (among other characterizations of semisimple algebras).

Theorem 2: For a finite dimensional algebra A TFAE:

1) A is semisimple.

2) A is isomorphic to the direct sum of some simple algebras.

Combining Theorem 1 & 2 we reduce the study of semisimple algebras to that of skew-fields. The further study of skew-fields will be the final topic for this class.

1.3) Structure of modules

By definition, all finite dimensional modules over a semisimple algebra, A, are completely reducible. So to understand how the modules look like, we need to classify the irreducibles.

Exercise: Let S be a skew-field and $n>0$. The $\mathrm{Mat}_n(S)$-module S^n (consisting of the column vectors v w. usual multiplication of a vector by a matrix) is irreducible (hint: use the same argument as in the proof that $\mathrm{Mat}_n(S)$ is a simple algebra.

Assuming Theorems 1&2, $A \simeq \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(S_i)$ for some $k, n_i>0$, and skew-fields S_i. Thx to the projection $A \twoheadrightarrow \mathrm{Mat}_{n_i}(S_i)$, we can view $S_i^{n_i}$ as an A-module. It's irreducible b/c the projection is surjective.

Theorem 3: Every irreducible A-module is isomorphic to exactly one of $S_i^{n_i}$'s.

2) Basics on modules.

Let A be a finite dimensional associative algebra over $\mathbb{F}$.

2.1) Terminology

We are interested in representations of A a.k.a. A-modules. We also refer to them as <u>left</u> modules: the data of a module, M, is a map $A\times M\longrightarrow M$, $(a,m)\mapsto am$, i.e. A acts on the left. We can also talk about <u>right</u> modules: these are vector spaces, N, w. a bilinear map $N\times A\longrightarrow N$, $(n,a)\mapsto na$, satisfying associativity, $(na)b=n(ab)$, and unit, $n1=n$, axioms.

Note that a right A-module is the same thing as a left A^{opp}-module, where, recall, Sec 3 of Lec 7, $A^{opp}=A$ as $\mathbb{F}$-vector space but with opposite multiplication: $a\cdot^{opp}b=ba$.

2.2) Regular module

The algebra A is both left & right module over itself w.r.t. multiplication. The left A-module A is usually called regular.

Examples: 1) Let $A = \mathrm{Mat}_n(S)$. Then $A \simeq (S^n)^{\oplus n}$ as A-modules – this just says that we can write an $n\times n$-matrix as a collection of its columns.

2) From 1) we see that for $A = \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(S_i)$, we get $A \simeq \bigoplus_{i=1}^{k} (S_i^{n_i})^{\oplus n_i}$ as A-modules.

A submodule in a left (resp. right) A-module A is called a left (resp., right) ideal. Note that a two-sided ideal (Sec 1.1) is the same thing as a left ideal, which is also a right ideal.

The following lemma partially describes a distinguished role that the regular module plays in the study of general & irreducible modules.

Lemma: 1) Let V be a finite dimensional A-module. Then V is isomorphic to a quotient of $A^{\oplus k}$ for some k.

2) Let V be an irreducible A-module. Then $\exists$ a maximal left ideal $I \subset A$ s.t. $V \simeq A/I$, an A-module isomorphism. Here "maximal" means: if $I' \subset A$ is a left ideal s.t. $I' \supsetneq I$, then $I' = A$.

Proof:

1: Let $v_1, \dots v_k$ be a spanning set for the A-module M (e.g. we can take a basis of the $\mathbb{F}$-vector space M). The map $A^{\oplus k} \xrightarrow{\varphi} M$, $(a_1, \dots a_k) \mapsto \sum_{i=1}^{k} a_i v_i$ is A-linear (exercise) & surjective – by the choice of $v_1, \dots v_k$. So $A^{\oplus k}/\ker\varphi \xrightarrow{\sim} M$.

2: For any $v \in M$, the subset $Av = \{av \mid a \in A\} \subset M$ is a submodule. Since M is irreducible, for $v \neq 0 \Rightarrow Av = M$ $\Rightarrow M \simeq A/I$, where $I = \ker[a \mapsto av]$ is a left ideal in A. The ideal I is maximal: $I'/I \subset A/I$ is a proper submodule of A/I for $I \subsetneq I' \subsetneq A$. $\square$

Corollary: TFAE:

(a) A is semisimple.

(b) The regular A-module is completely reducible.

Proof:

(a) $\Rightarrow$ (b) is a tautology. To prove (b) $\Rightarrow$ (a) recall that quotients & direct sums of completely reducible modules are completely reducible (Lemma in Sec 2.1 of Lec 5 proved in Lec 6). Now we use 1) of the previous lemma. $\square$

3) Bonus: an infinite dimensional simple algebra

By the 1st Weyl algebra over $\mathbb{F}$ we mean

$$W_1 := \mathbb{F}\langle x,y\rangle/(yx-xy-1),$$

where $\mathbb{F}\langle x,y\rangle$ we mean the algebra of noncommutative polynomials in x & y (a.k.a. the free algebra in x,y), it's basis are words in x & y and the product comes from concatenation. The algebra W_1 is simple iff $\operatorname{char}\mathbb{F}=0$.

Here are two (related) motivations for the relation $yx-xy=1$. The space $\mathbb{F}[x]$ becomes a module over W_1, where

$x \in W_1$ acts by the multiplication by x, while y acts by $\frac{d}{dx}$ (exercise). Also, up to rescaling, $yx - xy = 1$ is the commutation relation of the position & the momentum in Quantum Mechanics.