# Lecture 2: Basics, I.

0) Recap.

1) Example: from actions to representations, cont'd.

2) Constructions w. representations.

Bonus: More on "from actions to representations"

## 0) Recap.

In the first lecture we have defined a representation of a group, $G$, in a vector space, $V$, as a group homomorphism $\rho: G \to GL(V)$, where the target is the group of all invertible linear operators $V \to V$. Often, we abuse/abbreviate the terminology and say that $V$ itself is a representation of $G$ (w/o mentioning $\rho$ explicitly). Below we often write $g_V$ for $\rho(g)$. And as was mentioned in Sec. 1 of Lec 1, to give a representation of $G$ in $V$ is the same thing as to equip $V$ with a $G$-action by linear operators.

Let $X$ be a set equipped w. a $G$-action. From here we constructed a representation of $G$ in the vector space $\mathrm{Fun}(X,\mathbb{F})$ of functions $X\to\mathbb{F}$ (w. pointwise operations) by

$$[g.f](x)=f(g^{-1}x) \quad (f\in \mathrm{Fun}(X,\mathbb{F}),\ g\in G,\ x\in X).$$

Below we always consider $\mathrm{Fun}(X,\mathbb{F})$ w. this structure of a representation.

Our next task is to discuss this example in more detail.

## 1) Example: from actions to representations, cont'd.

First we want to understand how $G$ acts on certain vectors in $\mathrm{Fun}(X,\mathbb{F})$.

For $x\in X$, we can consider its "$\delta$-function" $\delta_x$ defined by

$$\delta_x(y)=\begin{cases}1, & x=y\\ 0, & \text{else}\end{cases}$$

Exercise: We have $g.\delta_x=\delta_{gx}$.

Rem: If $X$ is finite, then the functions $\delta_x$, $x\in X$, form a basis in $\mathrm{Fun}(X,\mathbb{F})$ – this is one reason why we care.

Now we consider some special cases.

Example 1: Let $G = S_n$, the symmetric group on $\{1,2,\dots,n\}$ & $X = \{1,2,\dots,n\}$ w. $G$ acting by permutations. The basis $\delta_1, \dots \delta_n$ identifies $\mathrm{Fun}(X, \mathbb{F}) \xrightarrow{\sim} \mathbb{F}^n$ and we get the representation of $S_n$ in $\mathbb{F}^n$ given by

$$g.(a_1, \dots, a_n) = (a_{g^{-1}(1)}, \dots, a_{g^{-1}(n)}), \; g \in S_n, \; a_i \in \mathbb{F}.$$

It is called the permutation representation.

Example 2 Now let $G$ be a finite group, and $X = G$ w. action by left multiplications: $g.h = gh$, so that, on the basis $\delta_h \in \mathrm{Fun}(G, \mathbb{F})$ $(h \in G)$, $G$ acts as $g.\delta_h = \delta_{gh}$. The resulting representation is called regular, it plays an important role in the theory.

Side remarks:

1) Students who took MATH 380 could observe that $\mathrm{Fun}(\bullet, \mathbb{F})$ is a contravariant functor from the category of

sets to the category of $\mathbb{F}$-vector spaces (and, even stronger to the category of commutative algebras). The construction of the $G$-representation in $\mathrm{Fun}(X,\mathbb{F})$ is a formal consequence of the functor claim.

2) Our usual setting in this course is that $G$ & $X$ are finite. A more interesting (and important) setting is when both $G$ & $X$ are infinite and come w. additional structure - and we can restrict the class of functions on $X$ we consider. We'll discuss this in the bonus section 3.

## 2) Constructions w. representations.

**2.1) Direct sums.** This has been mentioned in Sec 2 of Lec 1: if $V_1, \dots V_k$ are representations of $G$, then $V_1 \oplus \dots \oplus V_k = \{(v_1, \dots v_k) \mid v_i \in V_i\}$ is a representation of $G$ with $g.(v_1,\dots,v_k) := (gv_1,\dots,gv_k)$. To check this is indeed a representation is left as an exercise.

## 2.2) Subrepresentations.

**Definition:** Let $V$ be a representation of $G$. A subspace $U\subset V$ is called a subrepresentation if it is $G$-stable: $g_V(U)\subset U\ \forall g\in G$.

Note that the map $g\mapsto g_U:=g_V|_U: G\to GL(U)$ is also a representation of $G$ (exercise). We always consider $U$ w. this structure of a representation.

**Examples:** 1) Let $V_1, V_2$ be representations of $G$. Then $\{(v_1,0)|v_1\in V_1\}$, $\{(0,v_2)|v_2\in V_2\}$ are subrepresentations of $V_1\oplus V_2$. Note that they are naturally identified w. $V_1, V_2$ and this is how we view $V_1, V_2$ as subrepresentations of $V_1\oplus V_2$.

2) Let $X$ be a set acted on by $G$. The subspace of constant functions, $Fun_{const}(X,\mathbb{F})\subset Fun(X,\mathbb{F})$ is a subresentation of $Fun(X,\mathbb{F})$.

Now assume $X$ is finite. Consider

$$Fun_0(X,\mathbb{F})=\{f: X\to\mathbb{F}|\sum_{x\in X}f(x)=0\}\subset Fun(X,\mathbb{F}).$$

It's a subrepresentation: $G$ just permutes summands in $\sum_{x\in X}f(x)$.

3) Let $V$ be an arbitrary representation of $G$. An element $v \in V$ is called *G-invariant* if $g.v = v \ \forall g \in G$. The subset of all $G$-invariants is a subspace — e.g. $g.v_i = v_i$, $i = 1,2 \Rightarrow g.(v_1 + v_2) = [g$ acts by a linear operator$] = g.v_1 + g.v_2 = v_1 + v_2$. This subspace is denoted by $V^G$. It is a subrepresentation. Note that $\mathrm{Fun}_{const}(X, \mathbb{F}) \subset \mathrm{Fun}(X, \mathbb{F})^G$ w. equality $\Leftrightarrow X$ is an orbit.

Exercise: If $U, U' \subset V$ are subresentations, then so are $U \cap U'$, $U + U'$.

Remark: Suppose $\dim V = n$ and choose a basis $v_1, \dots v_n$ of $V$ s.t. $v_1, \dots v_k$ is a basis of $U$. The condition that $g_V(U) \subset U$ means that the matrix of $g_V$ in this basis is of block-triangular form:

$$k\{ \begin{pmatrix} A_g & B_g \\ \underbrace{0}_{k} & D_g \end{pmatrix}$$

Then in the basis $v_1, \dots v_k$ of $U$, the operator $g_U$ is given by the matrix $A_g$.

## 2.3) Quotient representations.

We are not going to see them often, but let's cover them for the sake of completeness.

Let $V$ be a vector space over $\mathbb{F}$, and $U \subset V$ be a subspace. We can form the quotient vector space, $V/U$, whose elements are the subsets of the form $v+U=\{v+u \mid u\in U\}$ for $v\in V$ and the operations are as follows:

$(v_1+U)+(v_2+U)=(v_1+v_2)+U$, $a(v_1+U)=av_1+U$, $v_1,v_2\in V$, $a\in\mathbb{F}$.

Exercise: Suppose now $V$ is a representation of $G$ & $U$ is a subrepresentation. Show that there is the unique $G$-representation in $V/U$ s.t.

$$g.(v+U)=g.v+U \quad \forall g\in G,\ v\in V.$$

Remark: We use the notation of the previous remark. Note that vectors $v_{k+1}+U,\dots,v_n+U$ form a basis in $V/U$. In this basis the operator $g_{V/U}$ is given by the matrix $D_g$.

So once we know the representations in $U, V/U$, we get partial info about $V$ - we know $A_g$ and $D_g$ (but not $B_g$). We'll see below that Maschke's Thm mentioned in Lec 1, Sec 2, can be stated as follows: under its assumptions - $\operatorname{char} \mathbb{F}=0$, $|G|<\infty$, we can choose $v_1, \dots v_n$ s.t. $B_g=0$, $\forall g \in G$.

## 2.4) Pullbacks under group homomorphisms.

Let $\varphi: H \to G$ be a group homomorphism and $V$ be a representation of $G$ via $\rho: G \to GL(V)$. Then we can view $V$ as a representation of $H$ via $\rho \circ \varphi: H \to GL(V)$. We sometimes call the resulting representation of $H$ the pullback (of $V$ to $H$).

Remark: One situation when we apply this construction is when $H$ is a subgroup of $G$ and $\varphi: H \to G$ is the inclusion. Then one talks about the restriction of representation of $G$ to $H$. One can try to understand the representations of $G$ via this techique (for suitable $H$). We'll see this in some homework problems.

## 3*) More on "from actions to representations"

The construction of a representation of $G$ in $\mathrm{Fun}(X,\mathbb{F})$ is useful to study representations of finite groups but not much beyond that - after all how interesting are functions on a finite set? However, the construction often applies when we consider (finite or infinite) group actions on infinite sets and look at certain nice functions. What we may want to do includes the following:

(I) Use the representation to understand the space of functions (Harmonic Analysis)

(II) Identify and study functions that play a special role for the representation of $G$ in $\mathrm{Fun}(X,\mathbb{F})$.

Below we will outline one example of (I) and two examples of (II).

## 2.1) Subalgebras of invariants.

This is an example of (II). Suppose that we are in the situation of Sec 1. Note that $f \in \mathrm{Fun}(X,\mathbb{F})^G$ iff $f$ is constant on $G$-orbits - from the definition of the $G$-action on $\mathrm{Fun}(X,\mathbb{F})$.

(*) So $x,y \in X$ are in the same $G$-orbit iff

$$f(x)=f(y) \quad \forall f \in \mathrm{Fun}(X,\mathbb{F}).$$

In many situations we want to know when two points are in the same orbit, for example, in Linear algebra this occurs in the investigation of "canonical forms." However, (*) above is useless for such purposes: the arbitrary functions on an infinite set are out of control.

In the situations of interest for Linear algebra (such as the classification of square matrices up to conjugation) $X$ is a finite dimensional vector space and $G$ acts linearly. It makes sense to speak about "polynomial functions" on $X$. By definition, these are functions that are polynomials in the coordinates w.r.t. some basis. Note that the linear changes of variables send polynomials to polynomials, so this definition doesn't depend on the choice of a basis.

Denote the space of polynomial functions on $X$ by $\mathbb{F}[X]$, this is a subspace of $\mathrm{Fun}(X,\mathbb{F})$. Moreover, it is a subrepresentation of $\mathrm{Fun}(X,\mathbb{F})$: precisely because any linear change of variables sends a polynomial to a polynomial. So one can consider the subspace $\mathbb{F}[X]^G$ of $G$-invariant polynomials. In order for the polynomial functions to be meaningful, assume $\mathbb{F}$ to be infinite.

*Example:* Let $G = S_n$ and $X$ be its permutation representation $\mathbb{F}^n$. Let $x_1, \dots x_n$ be the default coordinates on $X = \mathbb{F}^n$. The elements of $\mathbb{F}[X]^G$ are exactly the polynomials in $x_1, \dots x_n$ that do not change under any change of variables, i.e. the symmetric polynomials.

Representation theory greatly helps to study the subspace (in fact, the subalgebra) $\mathbb{F}[X]^G$. Assume, for simplicity, that $\operatorname{char} \mathbb{F} = 0$. Later in the course we will elaborate (as a bonus) on the following claim:

If $G$ is "reductive" (this includes all finite groups), then $\mathbb{F}[X]^G$ is finitely generated.

Remark: in general, it's no longer true that $f(x)=f(y)\ \forall f\in\mathbb{F}[X]^G \Rightarrow Gx=Gy$. For example, take $X=\mathbb{F}$ & $G=\mathbb{F}^\times$ (i.e $\mathbb{F}\setminus\{0\}$ w.r.t. multiplication) acting on $X$ via $g.x=gx$. In this case we have two orbits (zero & nonzero), while the only invariant polynomials are scalars (exercise). However if $G$ is finite, then the implication above is true, which is a somewhat harder exercise - on interpolation polynomials.

## 2.2) Harmonic analysis.

This is a manifestation of (I). Let's say $G$ acts on $X$ and we consider the corresponding representation in a space $\mathcal{F}$ that looks like the space of functions (for $X$ finite we just take $\mathcal{F}=\mathrm{Fun}(X,\mathbb{F})$ but for infinite $X$ we need to modify). We want to decompose an arbitrary function as a (finite or infinite) sum of "nice" functions.

Here is the most classical special case. Let $G=\{z\in\mathbb{C} \mid |z|=1\}$, this is a group w.r.t. multiplication. We can identify $G$ w. the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ via $x\in\mathbb{R}\mapsto e^{\sqrt{-1}x}$. Take $X=G$, $\mathbb{F}=\mathbb{C}$ and consider the action of $G$ on $X$ by multiplications.

Exercise: For $n\in\mathbb{Z}$, consider the function $x\mapsto e^{n\sqrt{-1}x}: X\to\mathbb{C}$. The 1-dimensional subspace spanned by this function is a subrepresentation of $\mathrm{Fun}(X,\mathbb{C})$.

We want to present an element of $\mathcal{F}$ (a space related to $\mathrm{Fun}(X,\mathbb{C})$) as a (possibly infinite) linear combination of the functions $e^{n\sqrt{-1}x}$. Of course, if the sum is infinite, it should converge in a suitable sense. Analysis comes into play and tells us that the best space to consider is $L^2(X)$ - Lebesgue square-integrable functions (or, more precisely, equivalence classes of such functions). Of course, the presentation of $f\in L^2(X)$ as an infinite linear combination of the functions $e^{n\sqrt{-1}x}$ is the Fourier expansion, one of the most fundamental techniques in Analysis.

A nice feature of the previous example that each summand $e^{n\sqrt{-1}x}$ is in its own irreducible representation and the representations for different $n$ are pairwise distinct. In other words, $L^2(X)$ decomposes as some kind of direct sum of pairwise distinct irreducible representations. This is known as a "multiplicity 1 result."

There is a number of other settings when we have such multiplicity 1 result. For example, we can take $G$ to be $SO_n(\mathbb{R})$, the group of orthogonal matrices w. det = 1. It acts on $\mathbb{R}^n$ preserving the unit sphere $\{(x_1,\dots x_n) \mid \sum x_i^2 = 1\}$. We take this sphere for $X$. The representation of $G$ in $L^2(X)$ is also multiplicity free.

This circle of questions was studied in MATH 628, Harmonic Analysis in F22.

## 2.3) Modular forms

Here is another important manifestation of (II)-modular forms. Our reference here is Serre "A Course in Arithmetic", Ch VII. Our $X$ here is the upper half-plane: $X = \{z \mid \mathrm{Im}\, z > 0\}$. It is acted on by the group $SL_2(\mathbb{R})$ of real $2\times 2$ matrices w. det $= 1$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} . z = \frac{az+b}{cz+d}$ (and is, actually, a single orbit). We take $G = SL_2(\mathbb{Z})$, the subgroup of $G$ of all matrices w. integral entries. Our $\mathcal{F}$ is the space of all holomorphic (= complex differentiable) functions. It is a representation of $SL_2(\mathbb{R})$ and hence of $G$.

We say that $f \in \mathcal{F}$ is weakly modular of weight $2k$ (in Serre's convention) if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Note that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, this condition translates to $f(z+1) = f(z)$ so $f$ can be expressed as a function of $q := e^{2\pi\sqrt{-1}\, z}$. If $f$ has limit as $q \to 0$, it's called a modular form.

Modular forms (and their generalizations such as automorphic forms) play a very important role in Number theory, Serre's book gives a brief intro to why. And their modern study (say, in Langlands program) is heavily based on Representation theory – but this far beyond this course.