Lecture 20: Finite dimensional associative algebras, II.

1) Partial proof of the three theorems from Lec 19

2) Modules over skew-fields.

3) What's next (slightly modified on 4/14).

1) Partial proof of the three theorems from Lec 19

1.0) Recap

Let $\mathbb{F}$ be a field. All $\mathbb{F}$-algebras in Section 1 are assumed to be finite dimensional & associative.

In Sec. 1 of Lec 19 we have stated the following:

I) Every simple $\mathbb{F}$-algebra is isomorphic to $\mathrm{Mat}_n(S)$ for unique n & skew-field S (fin. dim. $/\mathbb{F}$)

II) An $\mathbb{F}$-algebra is semisimple iff it's isomorphic to $\bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(S_i)$, where S_i are skew-fields.

III) The irreducible modules over $\bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(S_i)$ are exactly $S_i^{n_i}$, $i=1,\dots k$.

Our goal in this lecture are to prove some relatively easy partial statements.

Recall also that we've proved the following claims:

(A) An $\mathbb{F}$-algebra A is semisimple iff the regular A-module A is completely reducible. This is Corollary in Sec 2.2 of Lec 19.

(B) If $A = \bigoplus_{i=1}^{k} Mat_{n_i}(S_i)$, then $A \simeq \bigoplus_{i=1}^{k} (S^{n_i})^{\oplus n_i}$. This is Example in Sec 2.2 of Lec 19. Each $S_i^{n_i}$ is irreducible (see Sec 1.3 of Lec 19).

(C) The sub- & quotient modules & direct sums of completely reducible modules are completely reducible. This is Lemma in Sec 2.1 of Lec 5.

Corollary: $A = \bigoplus_{i=1}^{k} Mat_{n_i}(S_i)$ is a semisimple algebra.

Proof:

The regular A-module is the direct sum of irreducibles (by (B)) so completely reducible (by (C)). Now we use (A). $\square$

1.1) Goals.

Our goal is to prove the following weaker version of 1-3:

Theorem: The following claims are true:

1) Every simple algebra is semisimple and has a unique (up to iso) irreducible module.

2) The direct sum of simple algebras is semisimple.

3) The irreducible modules over $\bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(S_i)$ are exactly $S_i^{n_i}$, $i=1,\dots k$.

The proofs are going to be based on the following important result.

Proposition: Let A be an $\mathbb{F}$-algebra, $U_1,\dots U_k$ pairwise non-isomorphic irreducible A-modules. Let $\varphi_i : A \to \mathrm{End}(U_i)$ be the corresponding homomorphisms. Suppose $\bigcap_{i=1}^{k} \ker \varphi_i = \{0\}$. Then A is semisimple, and any irreducible A-module is isomorphic to one of U_i's.

1.2) Proof of Proposition

Consider the homomorphism $\varphi=(\varphi_1,\dots,\varphi_k): A \hookrightarrow \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$, so we can identify A w. $\operatorname{im}\varphi$ and view it as a subalgebra of $\bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$. In particular, $\bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$ becomes an A-module (w.r.t. multiplication on the left) and the regular module is a submodule. We have $\operatorname{End}_{\mathbb{F}}(U_i) \simeq U_i^{\oplus \dim U_i}$ as $\operatorname{End}(U_i)$-modules (compare to (B)). Hence $\bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i) \simeq \bigoplus_{i=1}^k U_i^{\oplus \dim U_i}$ as $\bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$-modules, and therefore as A-modules. But $\bigoplus_{i=1}^k U_i^{\oplus \dim U_i}$ is a completely reducible A-module (b/c the U_i's are irreducible). As a submodule in a completely reducible module, A is completely reducible, by (C), hence is a semisimple algebra by (A).

Also, A is isomorphic to the direct sum of some copies of U_i. It follows that if U is an irreducible A-module w. $U \not\simeq U_i \ \forall i$, then the multiplicity of U in A is 0. On the other had this multiplicity is $\dim U / \dim \operatorname{End}_A(U) > 0$ (Sec 2.1 in Lec 7). Contradiction. $\square$

1.3) Proof of Theorem

Proof of 1): Take an irreducible A-module U (e.g. minimal w.r.t. inclusion nonzero left ideal of A). This gives an algebra homomorphism $A \xrightarrow{\varphi} \mathrm{End}(U)$. Consider $\ker\varphi$. It's a two-sided ideal, different from A (b/c $\varphi(1)=1$). Since A is simple, $\ker\varphi=\{0\}$. Now we can apply Proposition to A & U to finish the proof. $\square$

Proof of 2): Let $A_1, \dots A_k$ be simple algebras & U_i be an irreducible A_i-module. Then $U_1, \dots U_k$ can be viewed as irreducible modules over $A=\bigoplus_{i=1}^k A_i$. Now apply Proposition to A and its modules $U_1,\dots,U_k$ $(A_i \hookrightarrow \mathrm{End}(U_i) \Rightarrow A \hookrightarrow \bigoplus_{i=1}^k \mathrm{End}(U_i))$.

Proof of 3): Apply Proposition to $A=\bigoplus_{i=1}^k \mathrm{Mat}_{n_i}(S_i)$ & $U_i=S_i^{n_i}$ for $i=1,\dots k$. Details are an exercise.

2) Modules over skew-fields.

To finish the proofs of (I)-(III) in Sec 1.0 we need to show that every (finite dimensional) simple (resp. semisimple) algebra A is isomorphic to $\mathrm{Mat}_n(S)$ - and n, S are determined uniquely (resp. A is isomorphic to $\bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(S_i)$). Let's explain how the proof for simple algebras works.

Let A be an algebra and U be its irreducible module. $\dim_{\mathbb{F}} A<\infty \Rightarrow \dim_{\mathbb{F}} U<\infty$: U is a quotient of A (Sec 2.2 of Lec 19)

Recall that by the Schur lemma (Lec 6), $\mathrm{End}_A(U)$ is a skew-field. The space U becomes a left module over $\mathrm{End}_A(U)$, hence a right module over $S:=\mathrm{End}_A(U)^{opp}$. The actions of A & S on U commute, in particular, the image of A in $\mathrm{End}_{\mathbb{F}}(U)$ lies in $\mathrm{End}_S(U)$.

Now suppose A is simple so that $A \hookrightarrow \mathrm{End}_S(U)$. We will see that:

(i) $A \xrightarrow{\sim} \mathrm{End}_S(U)$

(ii) $\mathrm{End}_S(U) \simeq \mathrm{Mat}_n(S)$ for $n:=\dim U/\dim S$, an isomorphism of algebras and S is recovered as $\mathrm{End}_A(U)^{opp}$.

(i) is more subtle and will be addressed in a later lecture. We'll now work towards (ii). This (and other things) require understanding modules over skew-fields.

2.1) Structure of modules.

The slogan is: modules over skew-fields behave just like vector spaces over fields. For example:

Lemma: Let S be a skew-field and M be a finitely generated module over S. Then M is a "free module" meaning that $M \xrightarrow{\sim} S^n$ for some $n \geq 0$. The number n is uniquely determined from M.

Proof: Let $m_1, \dots m_n \in M$ be generators: $\forall m \in M\ \exists s_1, \dots s_n \in S \mid m = \sum_{i=1}^{n} s_i m_i$. Assume n is minimal possible. Suppose we have a linear relation $\sum_{i=1}^{n} a_i m_i = 0$ for some $a_i \in S$. If $a_j \neq 0$, then we can invert it and express m_j via the remaining m_i. Contradiction w. n being minimal possible. It follows that $m_1, \dots m_n$ is a basis $\rightsquigarrow S^n \xrightarrow{\sim} M$.

To show that n is uniquely recovered from $M \Leftrightarrow$ $[S^n \xrightarrow{\sim} S^{n'} \Rightarrow n = n']$ we can argue as in the case of fields. Or, in the case $\dim_{\mathbb{F}} S < \infty$, the only case of interest for us, we observe that $\dim_{\mathbb{F}} S^n = n \dim_{\mathbb{F}} S$, which recovers n. $\square$

We call n from the lemma, the dimension of M over S, & write $\dim_S M$. If $\dim_{\mathbb{F}} S<\infty$, then $\dim_S M = \dim_{\mathbb{F}} M/\dim_{\mathbb{F}} S$.

Exercise: Let $\dim_S M = n$. Show that any $n+1$ elements in M are linearly dependent, and hence any n linearly independent elements form a basis.

2.2) Endomorphisms.

We now check (ii) mentioned above. Suppose S is a skew-field (and a finite dimensional $\mathbb{F}$-algebra for simplicity). Let M be a right S-module of dimension n.

Lemma: A choice of a basis in M identifies $\operatorname{End}_S(M)$ w. $\operatorname{Mat}_n(S)$.

Proof:

Let $m_1, \dots m_n$ be a basis in M yielding an identification $M \xrightarrow{\sim} S^n$. As usual, we view S^n as the space of column vectors. As discussed in Sec 1.3 of Lec 19, an element of $\operatorname{Mat}_n(S)$ gives an $\mathbb{F}$-linear endomorphism of S^n (multiplication of a column by a matrix). It is S-linear for the action of S on S^n by multiplications from <u>the right</u>. This gives an inclusion of $\mathbb{F}$-algebras $\mathrm{Mat}_n(S) \hookrightarrow \mathrm{End}_S(S^n)$.

Now, show that $\mathrm{Mat}_n(S) \xrightarrow{\sim} \mathrm{End}_S(S^n)$. Let $\varphi \in \mathrm{End}_S(S^n)$. So $\varphi(\sum_{i=1}^n e_i s_i) = \sum_{i=1}^n \varphi(e_i) s_i$. Then φ is given by the multiplication by the matrix with columns $\varphi(e_1), \ldots, \varphi(e_n)$. $\square$

2.3) Examples we care about.

Let A be a finite dimensional $\mathbb{F}$-algebra, U, V be finite dimensional A-modules w. U irreducible. So $S := \mathrm{End}_A(U)^{opp}$ is a skew-field. The space U itself is a right S-module and this is one of the modules we care about. Another module is $\mathrm{Hom}_A(U,V)$. It is a right $\mathrm{End}_A(U)$-module via $\varphi\alpha := \varphi \circ \alpha$ $(\varphi \in \mathrm{Hom}_A(U,V), \alpha \in \mathrm{End}_A(U))$ and hence a left S-module.

3) What's next?

Now we explain our approach to (a more general version of) i). The following will be proved later (likely in Lec 22). This statement is known as "Density Theorem".

Theorem: Let A be an associative algebra and $U_1, \dots U_k$ be its pairwise nonisomorphic finite dimensional irreducibles. Let $S_i := \mathrm{End}_A(U_i)^{opp}$ and let $\varphi_i : A \to \mathrm{End}_{\mathbb{F}}(U_i)$ be the homomorphism corresponding to the A-module U_i. Then the image of $(\varphi_1, \dots \varphi_k)$ is $\bigoplus_{i=1}^{k} \mathrm{End}_{S_i}(U_i)$.

Let's explain how our proof is going to go. As a left A-module $\bigoplus_{i=1}^{k} \mathrm{End}_{S_i}(U_i)$ is completely reducible. Indeed, $\bigoplus_{i=1}^{k} \mathrm{End}_{S_i}(U_i) \subset \bigoplus_{i=1}^{k} \mathrm{End}_{\mathbb{F}}(U_i) \simeq \bigoplus_{i=1}^{k} U_i^{\oplus \dim U_i}$ as A-modules.

We'll later see that one can give a description of all submodules in completely reducible modules. Applying this description to $A \subset \bigoplus_{i=1}^{k} \mathrm{End}_{S_i}(U_i)$ as a submodule for the left A-module structure, we'll deduce the Density Theorem from here with a little trick.