Lecture 20: Finite dimensional associative algebras, II.

1) Partial proof of the three theorems from Lec 19
2) Modules over skew fields
3) What’s next (slightly modified on 4/19)

1) Partial proof of the three theorems from Lec 19

1.0) Recap

Let $F$ be a field. All $F$-algebras in Section 1 are assumed to be finite dimensional & associative.

In Sec. 1 of Lec 19 we have stated the following:

I) Every simple $F$-algebra is isomorphic to $\text{Mat}_n(S)$ for unique $n$ & skew-field $S$ (fin. dim. $/F$)

II) An $F$-algebra is semisimple iff it’s isomorphic to $\bigoplus_{i=1}^k \text{Mat}_{n_i}(S_i)$, where $S_i$ are skew-fields.

III) The irreducible modules over $\bigoplus_{i=1}^k \text{Mat}_{n_i}(S_i)$ are exactly $S_i^{n_i}, i = 1 \ldots k$. 
Our goal in this lecture are to prove some relatively easy partial statements. Recall also that we’ve proved the following claims:

(A) An \( \mathbb{F} \)-algebra \( A \) is semisimple iff the regular \( A \)-module \( A \) is completely reducible. This is Corollary in Sec 2.2 of Lec 19.

(B) If \( A = \bigoplus_{i=1}^{k} \text{Mat}_{n_i}(S_i) \), then \( A \cong \bigoplus_{i=1}^{k} (S_i^n) \oplus S_i^n \). This is Example in Sec 2.2 of Lec 19. Each \( S_i^n \) is irreducible (see Sec 1.3 of Lec 19).

(C) The sub- & quotient modules & direct sums of completely reducible modules are completely reducible. This is Lemma in Sec 2.1 of Lec 5.

**Corollary:** \( A = \bigoplus_{i=1}^{k} \text{Mat}_{n_i}(S_i) \) is a semisimple algebra.

**Proof:**

The regular \( A \)-module is the direct sum of irreducibles (by (B)) so completely reducible (by (C)). Now we use (A). \( \square \)
1.1) Goals.

Our goal is to prove the following weaker version of 1-3:

**Theorem:** The following claims are true:

1) Every simple algebra is semisimple and has a unique (up to iso) irreducible module.

2) The direct sum of simple algebras is semisimple.

3) The irreducible modules over $\bigoplus_{i=1}^{k} \text{Mat}_n(S_i)$ are exactly $S_i^n$, $i = 1, \ldots, k$.

The proofs are going to be based on the following important result.

**Proposition:** Let $A$ be an $\mathbb{F}$-algebra, $U_1, \ldots, U_k$ pairwise non-isomorphic irreducible $A$-modules. Let $\varphi_i : A \to \text{End}(U_i)$ be the corresponding homomorphisms. Suppose $\bigcap_{i=1}^{k} \ker \varphi_i = \{0\}$. Then $A$ is semisimple, and any irreducible $A$-module is isomorphic to one of $U_i$'s.
1.2) Proof of Proposition

Consider the homomorphism ψ = (ψ₁, ..., ψₖ): A → \bigoplus_{i=1}^{k} \text{End}_F(U_i), so we can identify A w. i. m. im ψ and view it as a subalgebra of \bigoplus_{i=1}^{k} \text{End}_F(U_i).

In particular, \bigoplus_{i=1}^{k} \text{End}_F(U_i) becomes an A-module (w.r.t. multiplication on the left) and the regular module is a submodule. We have \text{End}_F(U_i) \cong U_i^\oplus \dim U_i as \text{End}(U_i)-modules (compare to (B)). Hence \bigoplus_{i=1}^{k} \text{End}_F(U_i) \cong \bigoplus_{i=1}^{k} U_i^\oplus \dim U_i as \bigoplus_{i=1}^{k} \text{End}_F(U_i)-modules, and therefore as A-modules. But \bigoplus_{i=1}^{k} U_i^\oplus \dim U_i is a completely reducible A-module (b/c the U_i's are irreducible). As a submodule in a completely reducible module, A is completely reducible, by (C), hence is a semisimple algebra by (A).

Also, A is isomorphic to the direct sum of some copies of U_i. It follows that if U is an irreducible A-module w. U \neq U_i \; \forall \; i, then the multiplicity of U in A is 0. On the other hand this multiplicity is \dim U/\dim \text{End}_A(U) > 0 (Sec 2.1 in Lec7).

Contradiction. □
13) Proof of Theorem

Proof of 1): Take an irreducible $A$-module $U$ (e.g. minimal w.r.t. inclusion nonzero left ideal of $A$). This gives an algebra homomorphism $A \rightarrow \text{End}(U)$. Consider ker $q$. It's a two-sided ideal, different from $A$ ($q(1) = 1$). Since $A$ is simple, ker $q = \{0\}$. Now we can apply Proposition to $A \otimes U$ to finish the proof. □

Proof of 2): Let $A_1, \ldots, A_k$ be simple algebras & $U_i$ be an irreducible $A_i$-module. Then $U_1, \ldots, U_k$ can be viewed as irreducible modules over $A = \bigoplus_{i=1}^k A_i$. Now apply Proposition to $A$ and its modules $U_1, \ldots, U_k$ ($A_i \rightarrow \text{End}(U_i) \Rightarrow A \rightarrow \bigoplus_{i=1}^k \text{End}(U_i)$).

Proof of 3): Apply Proposition to $A = \bigoplus_{i=1}^k \text{Mat}_n(S_i)$ & $U_i = S_i^{m_i}$ for $i = 1, \ldots, k$. Details are an exercise.

2) Modules over skew fields

To finish the proofs of (I) - (III) in Sec 1.0 we need to show that every (finite dimensional) simple (resp. semisimple)
algebra $A$ is isomorphic to $\text{Mat}_n(S)$ and $n, S$ are determined uniquely (resp. $A$ is isomorphic to $\bigoplus_{i=1}^k \text{Mat}_{n_i}(S_i)$). Let's explain how the proof for simple algebras works.

Let $A$ be an algebra and $U$ be its irreducible module.

$\dim_F A < \infty \Rightarrow \dim_F U < \infty$: $U$ is a quotient of $A$ (Sec 2.2 of Lec 19)

Recall that by the Schur lemma (Lec 6), $\text{End}_A(U)$ is a skew-field. The space $U$ becomes a left module over $\text{End}_A(U)$, hence a right module over $S := \text{End}_A(U)$. The actions of $A$ and $S$ on $U$ commute, in particular, the image of $A$ in $\text{End}_F(U)$ lies in $\text{End}_S(U)$.

Now suppose $A$ is simple so that $A \hookrightarrow \text{End}_S(U)$. We will see that:

(i) $A \sim \text{End}_S(U)$

(ii) $\text{End}_S(U) \sim \text{Mat}_n(S)$ for $n := \dim U / \dim S$, an isomorphism of algebras and $S$ is recovered as $\text{End}_A(U)$. This is more subtle and will be addressed in a later lecture. We'll now work towards (ii). This (and other things) require understanding modules over skew-fields.
2.1) Structure of modules.

The slogan is: modules over skew-fields behave just like vector spaces over fields. For example:

**Lemma:** Let $S$ be a skew-field and $M$ be a finitely generated module over $S$. Then $M$ is a "free module" meaning that $M \cong S^n$ for some $n > 0$. The number $n$ is uniquely determined from $M$.

**Proof:** Let $m_1, \ldots, m_n \in M$ be generators: $\forall m \in M \exists s_1, \ldots, s_n \in S | m = \sum_{i=1}^n s_i m_i$. Assume $n$ is minimal possible. Suppose we have a linear relation $\sum_{i=1}^n a_i m_i = 0$ for some $a_i \in S$. If $a_j \neq 0$, then we can invert it and express $m_j$ via the remaining $m_i$. Contradiction w. $n$ being minimal possible. It follows that $m_1, \ldots, m_n$ is a basis $\rightarrow S^n \hookrightarrow M$.

To show that $n$ is uniquely recovered from $M$ $\Rightarrow$ $\left[ S^n \hookrightarrow S^n' \Rightarrow n = n' \right]$ we can argue as in the case of fields. Or, in the case $\dim_F S < \infty$, the only case of interest for us, we observe that $\dim_F S^n = n \cdot \dim_F S$, which recovers $n$. $\square$
We call \( n \) from the lemma, the dimension of \( M \) over \( S \), & write \( \dim_S M \). If \( \dim_F S < \infty \), then \( \dim_S M = \dim_F M / \dim_F S \).

**Exercise**: Let \( \dim_S M = n \). Show that any \( n+1 \) elements in \( M \) are linearly dependent, and hence any \( n \) linearly independent elements form a basis.

2.2) **Endomorphisms**.

We now check (ii) mentioned above. Suppose \( S \) is a skew-field (and a finite dimensional \( \mathbb{F} \)-algebra for simplicity). Let \( M \) be a right \( S \)-module of dimension \( n \).

**Lemma**: A choice of a basis in \( M \) identifies \( \text{End}_S(M) \) w \( \text{Mat}_n(S) \).

**Proof**: Let \( m_1, \ldots, m_n \) be a basis in \( M \) yielding an identification \( M \cong S^n \). As usual, we view \( S^n \) as the space of column vectors. As discussed in Sec 1.3 of lec 19, an element of \( \text{Mat}_n(S) \) gives an \( \mathbb{F} \)-linear endomorphism of \( S^n \) (multiplication of a
column by a matrix). It is $S$-linear for the action of $S$ on $S^n$ by multiplications from the right. This gives an inclusion of \( \mathbb{F}\)-algebras $\text{Mat}_n(S) \hookrightarrow \text{End}_S(S^n)$. 

Now, show that $\text{Mat}_n(S) \hookrightarrow \text{End}_S(S^n)$. Let $\varphi \in \text{End}_S(S^n)$. So $\varphi(\sum_{i=1}^n e_i s_i) = \sum_{i=1}^n \varphi(e_i) s_i$. Then $\varphi$ is given by the multiplication by the matrix with columns $\varphi(e_1), \ldots, \varphi(e_n)$. \( \square 

2.3) \text{Examples we care about.} 

Let $A$ be a finite dimensional $\mathbb{F}$-algebra, $U, V$ be finite dimensional $A$-modules with $U$ irreducible. So $S := \text{End}_A(U)^{\text{opp}}$ is a skew-field. The space $U$ itself is a right $S$-module and this is one of the modules we care about. Another module is $\text{Hom}_A(U, V)$. It is a right $\text{End}_A(U)$-module via $\varphi \cdot \alpha := \varphi \circ \alpha$ ($\varphi \in \text{Hom}_A(U, V), \alpha \in \text{End}_A(U)$) and hence a left $S$-module.

3) What’s next? 

Now we explain our approach to (a more general version of) i). The following will be proved later (likely in Lec 22). This
Theorem: Let $A$ be an associative algebra and $U_1, \ldots, U_k$ be its pairwise nonisomorphic finite dimensional irreducibles. Let $S_i := \text{End}_A(U_i)^{op}$ and let $q_i : A \rightarrow \text{End}_E(U_i)$ be the homomorphism corresponding to the $A$-module $U_i$. Then the image of $(q_1, \ldots, q_k)$ is $igoplus_{i=1}^k \text{End}_{S_i}(U_i)$.

Let’s explain how our proof is going to go. As a left $A$-module $\bigoplus_{i=1}^k \text{End}_{S_i}(U_i)$ is completely reducible. Indeed, $\bigoplus_{i=1}^k \text{End}_{S_i}(U_i) < \bigoplus_{i=1}^k \text{End}_E(U_i) \simeq \bigoplus_{i=1}^k U_i^{\oplus \dim U_i}$ as $A$-modules.

We’ll later see that one can give a description of all submodules in completely reducible modules. Applying this description to $A \subset \bigoplus_{i=1}^k \text{End}_{S_i}(U_i)$ as a submodule for the left $A$-module structure, we’ll deduce the Density Theorem from here with a little trick.