Lecture 21: Finite dimensional associative algebras, II.

0) Goal

1) Tensor products over algebras

2) Canonical decomposition into irreducibles.

Ref: [E], Sec 3.1.

0) Goal

In Sec 3 of Lec 20 we've stated the Density theorem. We've mentioned that this requires describing submodules in finite dimensional completely reducible modules. This in turn will require writing a "canonical decomposition" into irreducibles. For this we will need tensor products of modules over algebras.

1) Tensor products over algebras

1.1) Case of general algebras

Let B be an (associative unital) algebra over a field $\mathbb{F}$. Let M be a right B-module & N be a left B-module. We want to define $M \otimes_B N$ and study its basic properties.

For simplicity, assume $\dim_{\mathbb{F}} M, \dim_{\mathbb{F}} N < \infty$.

Form the tensor product $M \otimes_{\mathbb{F}} N$. Inside consider the $\mathbb{F}$-subspace $K_{M,N} := \mathrm{Span}_{\mathbb{F}}(mb \otimes n - m \otimes bn \mid b \in B, m \in M, n \in N)$. Set

$$M \underset{B}{\otimes} N := M \otimes_{\mathbb{F}} N / K_{M,N},$$ this is an $\mathbb{F}$-vector space.

The image of $m \otimes n$ in $M \otimes_B N$ ($m \in M, n \in N$) will still be denoted by $m \otimes n$.

Here's a universal property generalizing that in Lec 4.

Lemma: Let W be a vector space and $\beta: M \times N \to W$ be an $\mathbb{F}$-bilinear map s.t.

(1) $\quad \beta(mb, n) = \beta(m, bn) \quad \forall b \in B, m \in M, n \in N.$

Then $\exists!$ $\mathbb{F}$-linear map $\hat{\beta}: M \otimes_B N \to W$ s.t. $\hat{\beta}(m \otimes n) = \beta(m,n)$

Proof: Let $\tilde{\beta}: M \otimes_{\mathbb{F}} N \to W$ be the unique $\mathbb{F}$-linear map s.t. $\tilde{\beta}(m \otimes n) = \beta(m,n)$. (1) translates to $\tilde{\beta}(mb \otimes n - m \otimes bn) = 0, \forall b, m, n \Leftrightarrow$ $\tilde{\beta}(K_{M,N}) = \{0\}$, hence uniquely factors through $\hat{\beta}$. $\square$

Now we establish some other properties of $M \otimes_B N$.

2

Exercise: Let M_1, M_2 be left B-modules & N be a right B-module. Show that, under the identification (Sec 1.4 of Lec 4)

$$(M_1 \oplus M_2) \otimes_{\mathbb{F}} N \simeq M_1 \otimes_{\mathbb{F}} N \oplus M_2 \otimes_{\mathbb{F}} N$$

we have $K_{M_1 \oplus M_2, N} = K_{M_1, N} \oplus K_{M_2, N}$. Deduce a natural isomorphism

$$(M_1 \oplus M_2) \otimes_B N \simeq M_1 \otimes_B N \oplus M_2 \otimes_B N,$$

Similarly, show that $M \otimes_B (N_1 \oplus N_2) \simeq M \otimes_B N_1 \oplus M_2 \otimes_B N$.

Example: We identify $B \otimes_B N$ w. N. To produce an $\mathbb{F}$-linear map $B \otimes_B N \to N$ consider the $\mathbb{F}$-bilinear map $B \times N \to N$, $(b, n) \mapsto bn$ and observe that it satisfies (1) giving the unique linear map $B \otimes_B N \to N$ w. $b \otimes n \mapsto bn$. Its inverse is given by $N \to B \otimes_B N$ $n \mapsto 1 \otimes n$. To check this is an inverse is an exercise.

Similarly, $M \otimes_B B \xrightarrow{\sim} M$ w. $m \otimes b \mapsto mb$.

Together w. Exercise, this example shows isomorphisms

(2) $\quad B^{\oplus n} \otimes_B N \xrightarrow{\sim} N^{\oplus n}, \quad M \otimes_B B^n \xrightarrow{\sim} M^{\oplus n}.$

Remark: Suppose A is another algebra. Suppose, further, that M is a left A-module so that the actions of A & B on M commute: $(am)b = a(mb)$ $\forall\, a\in A, b\in B, m\in M$. We claim that $M\otimes_B N$ has a unique A-module structure s.t. $a(m\otimes n) = (am)\otimes n$. Indeed, fix $a\in A$. Note that the map

$$\beta_a: M\times N \longrightarrow M\otimes_B N,\quad \beta_a(m,n) = (am)\otimes n$$

is $\mathbb{F}$-bilinear & satisfies (1) in Lemma leading to an $\mathbb{F}$-linear map $\hat\beta_a: M\otimes_B N \longrightarrow M\otimes_B N$.

Exercise: 1) $a\mapsto \hat\beta_a: A \longrightarrow End_{\mathbb{F}}(M\otimes_B N)$ is an algebra homomorphism (so $M\otimes_B N$ is indeed an A-module)

2) If W in Lemma is an A-module & β is A-linear, then $\hat\beta$ is A-linear too.

1.2) Case of skew-fields.

Let S be a skew-field, and let M & N be right & left S-modules. Assume S is a finite dimensional algebra over $\mathbb{F}$ & $\dim_S M, \dim_S N<\infty$. In this case one can write a basis in $M\otimes_S N$ as follows. Pick an S-basis $n_1,\ldots n_\ell\in N$

and an $\mathbb{F}$-basis $m_1, \dots m_k \in M$.

Lemma: The elements $m_i \otimes n_j$, $i = 1, \dots k$, $j = 1, \dots, \ell$, form an $\mathbb{F}$-basis in $M \otimes_S N$.

Sketch of proof: The choice of $n_1, \dots n_\ell$ gives an identification $N \xrightarrow{\sim} S^\ell$. Now we use isomorphism (2). Details are left as an exercise. $\square$

One has a direct analog of this for an S-basis in M & $\mathbb{F}$-basis in N.

Another special feature of skew-fields is that "tensoring preserves submodules." Let $N' \subset N$ be an S-submodule. We claim that $M \otimes_S N'$ can be viewed as a subspace in $M \otimes_S N$.

Exercise: 1) $\exists!$ $\mathbb{F}$-linear map $M \otimes_S N' \longrightarrow M \otimes_S N$ sending $m \otimes n'$ ($m \in M$, $n' \in N$) to $m \otimes n'$ (where n' is now viewed as an element of N).

2) It's injective: Show that if it's injective for right S-modules M_1, M_2, then it's injective for $M_1 \oplus M_2$. So the problem reduces to $M=S$, which can be handled using Example in Sec 1.1.

1.3) Homs & tensors.

Recall, Sec 1.3 of Lec 4, that if U, V are finite dimensional vector spaces, then we have a natural isomorphism

$$V^* \otimes_{\mathbb{F}} U \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}}(V, U).$$

This generalizes to modules over skew-fields as follows. Let U, V be finite dimensional left modules over S. We can form the $\mathbb{F}$-vector space $\mathrm{Hom}_S(U,V)$.

Set $V^* = \mathrm{Hom}_S(V, S)$ (homomorphisms of left S-modules). V^* is a right S-module via $[\alpha s](v) = \alpha(v)s$, $\alpha \in V^*, s \in S, v \in V$ (exercise). We can also identify U with $\mathrm{Hom}_S(S, U)$ via $\varphi \in \mathrm{Hom}_S(S,U) \mapsto \varphi(1) \in U$ (the inverse map sends u to $s \mapsto su$). Then we have the composition map $V^* \times U \longrightarrow \mathrm{Hom}_S(U,V)$. It uniquely extends to an $\mathbb{F}$-linear map (exercise)

(4) $$V^* \otimes_S U \longrightarrow \mathrm{Hom}_S(V, U)$$

(4) is an isomorphism: similarly to 2) of Exercise in Sec 1.2 we reduce to the case when $U=S$, where (4) becomes $\alpha\otimes s \mapsto \alpha s$, this is a special case of the identification from Example in Sec 1.1.

2) Canonical decomposition into irreducibles.

Let A be an associative algebra and V be its finite dimensional completely reducible module. Then there is an A-module isomorphism

(5) $$\bigoplus_{i=1}^{k} U_i^{\oplus m_i} \xrightarrow{\sim} V.$$

where $U_1, \dots U_k$ are pairwise nonisomorphic irreducibles, and $m_i = \dim_{\mathbb{F}} \operatorname{Hom}_A(U_i, V)/\dim_{\mathbb{F}} \operatorname{End}_A(U_i)$, see Sec 2 of Lec 6.

Isomorphism (5) is <u>not</u> unique, in general. For example, if $A=\mathbb{F}$, then (5) amounts to an isomorphism $\mathbb{F}^m \xrightarrow{\sim} V$, i.e., to a choice of basis in V. We want to come up with a canonical analog of (5).

Let $S_i := \operatorname{End}_A(U_i)^{opp}$, $M_i := \operatorname{Hom}_A(U_i, V)$. Then M_i is a left module over S_i (Sec 2.3 of Lec 20). Moreover,

$\dim_{S_i} M_i = \dim_{\mathbb{F}} M_i / \dim_{\mathbb{F}} S_i = m_i$

On the other hand, U_i is a right S_i-module so $U_i \otimes_{S_i} M_i$ makes sense. This is an A-module as explained in Remark in Sec 1.1. It has an A-linear map to V. Indeed, consider the $\mathbb{F}$-bilinear map $U_i \times M_i \xrightarrow{\beta} V$, $(u, \varphi) \mapsto \varphi(u)$, $u \in U_i$, $\varphi \in M_i = \mathrm{Hom}_A(U_i, V)$. It satisfies:

$$\beta(us, \varphi) = \varphi(s(u)) = (\varphi \circ s)(u) = \beta(u, s\varphi) \quad \forall s \in S_i \ \&$$

$$\beta(au, \varphi) = \varphi(au) = [\varphi \text{ is } A\text{-linear}] = a\,\varphi(u) = a\,\beta(u, \varphi).$$

By Lemma in Sec 1, $\exists!$ $\mathbb{F}$-linear $\hat{\beta}: U_i \otimes_{S_i} M_i \to V$ w. $\hat{\beta}(u \otimes \varphi) = \varphi(u)$ and, by Remark, it's A-linear. We write φ_i for β when we want to indicate the dependence on i.

Now consider the map

$$\varphi = (\varphi_1, \dots \varphi_k): \bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i \longrightarrow V$$

(that sends $(\alpha_1, \dots, \alpha_k) \in \bigoplus_i U_i \otimes_{S_i} M_i$ to $\sum_i \varphi_i(\alpha_i)$).

It's A-linear b/c the individual maps φ_i are.

Proposition: φ is an isomorphism.

Proof: Fix an identification (5): $V \simeq \bigoplus_{i=1}^{k} U_i^{\oplus m_i}$. For $j=1,\dots m_i$, consider the inclusion ι_{ij} of the jth copy of U_i into V. It's an A-linear map, hence an element of $M_i = \mathrm{Hom}_A(U_i, V)$.

We claim that, $\forall$ i, the elements ι_{ij}, $j=1,\dots m_i$, form an S_i-basis in M_i. Since $\dim_{S_i} M_i = m_i$, thx to Exercise in Sec 2.1 of Lec 20, it's enough to show that they are linearly independent. This follows b/c the image of ι_{ij} is in the jth copy of U_i inside V.

The choice of the S_i-basis ι_{ij} in M_i identifies $U_i \otimes_{S_i} M_i$ w. $U_i^{\oplus m_i}$. The composition $U_i^{\oplus m_i} \xrightarrow{\sim} U_i \otimes_{S_i} M_i \xrightarrow{\varphi_i} V$ on the jth copy of U_i is $u \mapsto u \otimes \iota_{ij} \mapsto \iota_{ij}(u)$. So the composition

$$\bigoplus_{i=1}^{k} U_i^{\oplus m_i} \xrightarrow{\sim} \bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i \xrightarrow{\varphi} V \xrightarrow[(5)]{\sim} \bigoplus_{i=1}^{k} U_i^{\oplus m_i}$$

is the identity finishing the proof. $\square$