Lecture 22: Finite dimensional associative algebras, IV.

1) Description of submodules

2) Proof of Density Theorem

Ref: [E], Secs 3.1, 3.2.

1) Description of submodules

1.1) Main result.

Let $\mathbb{F}$ be a field, $A$ be an associative $\mathbb{F}$-algebra. Let $U_1, \ldots, U_k$ be pairwise non-isomorphic finite dimensional irreducible $A$-modules, $S_i := \text{End}_A(U_i)^{op}$ (so that $U_i$ is also a right $S_i$-module and the $A$- and $S_i$-actions on $U_i$ commute).

Let $M_i, i = 1, \ldots, k$, be a finite dimensional left $S_i$-module.

So $\bigoplus_{i=1}^k U_i \otimes_{S_i} M_i$ becomes a left $A$-module (with $A$ acting on $U_i \otimes_{S_i} M_i$ via $a(u \otimes m) = au \otimes m$).

We start by constructing a family of $A$-submodules of $\bigoplus_{i=1}^k U_i \otimes_{S_i} M_i$. Let $N_i$ be an $S_i$-submodule of $M_i$. By Sec 1.2 of Lec 21, we can view $U_i \otimes_{S_i} N_i$ as an $\mathbb{F}$-subspace of $U_i \otimes_{S_i} M_i$. From the construction of the $A$-action on $U_i \otimes_{S_i} M_i$
we see that $U_i \otimes_{S_i} N_i$ is an $A$-submodule in $U_i \otimes_{S_i} M_i$.
So $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i$ is an $A$-submodule of $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i$.

**Proposition:** Every $A$-submodule $V' \subset V := \bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i$ has the form $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i$ for some $S_i$-submodules $N_i \subset M_i$.

**Proof:** Let’s discuss the idea first. We know that we have natural isomorphisms (Sec 2 of Lec 21)
\[
\psi: \bigoplus_{i=1}^{k} U_i \otimes_{S_i} \text{Hom}_{A}(U_i, V) \rightarrow V
\]
\[
\psi': \bigoplus_{i=1}^{k} U_i \otimes_{S_i} \text{Hom}_{A}(U_i, V') \rightarrow V'.
\]
Next, $\text{Hom}_{A}(U_i, V')$ embeds as an $S_i$-submodule into $\text{Hom}_{A}(U_i, V)$. The proof is then checking the details, and in particular matching various identifications.

**Step 1:** We claim that $M_i$ is identified w. $\text{Hom}_{A}(U_i, V)$. Namely, $\text{Hom}_{A}(U_i, V) = \left[ \text{Hom}_{A}(U_i, U_j) = 0 \text{ for } i \neq j \right] = \text{Hom}_{A}(U_i, U_i \otimes_{S_i} M_i)$. For $m \in M_i$ consider $\varphi_m: U_i \rightarrow U_i \otimes_{S_i} M_i$, $\varphi_m(u) = u \otimes m$, an $A$-linear map; $m \mapsto \varphi_m$ is an $S_i$-linear map.
\[ M_i \to \text{Hom}_A(U_i, U_i \otimes_{S_i} M_i), \text{ where } S_i = \text{End}_A(U_i)^{\text{opp}} \text{ acts on the target} \]
by taking compositions on the right, & it's injective (these are left as exercise, for the injectivity use a description of a basis in \( U_i \otimes_{S_i} M_i \), Sec 2.2 in Lec 21). Now note that if \( \dim_{S_i} M_i = n \), then \( U_i \otimes_{S_i} M_i \cong U_i^{\otimes n} \), so \( \dim \text{Hom}_A(U_i, U_i \otimes_{S_i} M_i) = n \dim \text{Hom}_A(U_i, U_i) = n \dim S_i = \dim Hom M_i. \)
So \( \imath \mapsto \varphi_{\imath}: M_i \rightarrow \text{Hom}_A(U_i, V). \)

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**Step 2:** Recall, Sec 2 of Lec 21, that for an arbitrary finite direct sum \( V \) of \( U_i \)'s, we have
\[ \psi: \bigoplus_{i=1}^{k} U_i \otimes_{S_i} \text{Hom}_A(U_i, V) \rightarrow V, \sum_{i=1}^{k} u_i \otimes \varphi_i \mapsto \sum_{i=1}^{k} \varphi_i(u_i) \]

Thus to Step 1, for \( V = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i \), we also have
\[ \xi: V \rightarrow \bigoplus_{i=1}^{k} U_i \otimes_{S_i} \text{Hom}_A(U_i, V), \sum_{i=1}^{k} u_i \otimes m_i \mapsto \sum_{i=1}^{k} u_i \otimes \varphi_{m_i} \]
Since \( \varphi_{m_i}(u) = u \otimes m_i \), we have \( \psi \xi = \text{id}_V \Rightarrow \xi = \psi'. \)

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**Step 3:** Consider \( N_i := \text{Hom}_A(U_i, V') = \{ \varphi \in \text{Hom}_A(U_i, V) | \imath \mapsto \varphi \in V' \} \)
This is a subspace in \( M_i = \text{Hom}_A(U_i, V). \) Moreover, it's an \( S_i \)-submodule: \( S_i = \text{End}_A(U_i)^{\text{opp}} \text{ acts on } \text{Hom}_A(U_i, V) \) by \( s \varphi = \varphi \circ s \), so
We claim that $V' = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i$, which will finish the proof. Indeed, let $\psi': \bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i \to V'$ be the analog of $\psi$ for $V'$. The following diagram commutes by the construction of $\psi$ (and $\psi'$):

\[
\begin{array}{ccc}
\bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i & \xrightarrow{\psi'} & V' \\
\downarrow & & \downarrow \\
\bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i & \xrightarrow{\psi} & V
\end{array}
\]

The vertical maps are inclusions, and we identify $M_i$ with $\text{Hom}_A(U_i, V)$ as in Step 1. In particular, we see that $V'$ is indeed of the form $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i$ for $S_i$-submodules $N_i \subset M_i$. \(\Box\)

**Exercise**: Show that $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} N'_i$ (for submodules $N_i, N'_i \subset M_i$) implies $N_i = N'_i$ for $i$ (hint: use bases in tensor products introduced in Sec 1.2 of Lec 21). This shows that $N_i$'s in Proposition are uniquely recovered from $V'$.

**Remark**: Thx to Sec 2 of Lec 21, Proposition describes submodules in an arbitrary completely reducible module.
2) Proof of Density Theorem.

2.1) Statement

The theorem was stated in Sec 3 of Lec 10.

Theorem: Let $A$ be an associative algebra and $U_1, \ldots, U_k$ be its pairwise nonisomorphic finite dimensional irreducibles. Let $S_i := \text{End}_A(U_i)_{\text{opp}}$ and let $\varphi_i : A \to \text{End}_F(U_i)$ be the homomorphism corresponding to the $A$-module $U_i$. Then the image of $(\varphi_1, \ldots, \varphi_k)$ is $\bigoplus_{i=1}^k \text{End}_{S_i}(U_i)$.

Before we get to the proof, let’s record

Corollary/Exercise: Suppose that $\text{dim}_F A < \infty$. Then the number of irreducible (automatically, finite dimensional) $A$-modules (up to iso) is $\leq \text{dim} A$.

2.2) Proof of Density Theorem

Again, let’s start w. an idea. We’ll identify $\bigoplus_{i=1}^k \text{End}_{S_i}(U_i)$
\[ \bigoplus_{i=1}^{n} U_i \otimes S_i U_i^* \] and then use Proposition to show that
\[ \text{im} \varphi = \bigoplus_{i=1}^{n} U_i \otimes S_i N_i \] for \( N_i \subset U_i^* \). If \( N_i \not\subset U_i^* \) for some \( i \), then, as we'll check, every element of im \( \varphi \) annihilates a nonzero vector in \( U_i \). This will give a contradiction.

**Step 1:** Analogously to Sec 13 of Lec 21, for a right \( S \)-module \( U \) (w. \( \dim_U U < \infty \)) we have an isomorphism

\[ (*) \quad U \otimes_S U^* \sim \text{End}_S(U). \]

Here \( U^* = \text{Hom}_S(U, S) \), we identify \( U \) w. \( \text{Hom}_S(S, U) \) via \( u \mapsto [s \mapsto us] \). Then \((*)\) is given by

\[ \psi \otimes u \mapsto \psi \circ u, \quad \psi \in \text{Hom}_S(S, U), \quad u \in \text{Hom}_S(U, S). \]

**Step 2:** Suppose now \( U \) is a left \( A \)-module in such a way that the actions of \( A \& S \) commute. So \( U \otimes_S U^* \) acquires a left \( A \)-module structure via \( a(u \otimes \alpha) = au \otimes \alpha \), \( \text{End}_S(U) \) is also a left \( A \)-module: via

\[ a \gamma = a \circ \gamma, \quad a \in A, \quad \gamma \in \text{End}_S(U). \]

It's left as an exercise to check that \((*)\) is an \( A \)-
module homomorphism.

Step 3: So, as an $A$-module,

$$\bigoplus_{i=1}^{k} \text{End}_{S_i}(U_i) = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i, \ M_i : = U_i^*.$$  

Note that $\text{Im} \varphi < \bigoplus_{i=1}^{k} \text{End}_{S_i}(U_i)$ is an $A$-submodule.

By Proposition in Sec 1, \(\exists \ N_i \subset M_i = U_i^* \) s.t.

$$\text{Im} \varphi = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i.$$  

We need to show that $N_i = M_i \not\supset i$. Assume the contrary: $N_i \subsetneq M_i$.

Step 4: We claim $\exists \ \upsilon \in U_i$ s.t. $\alpha(\upsilon) = 0 \ \forall \ \alpha \in N_i$.

For this, observe first that $U_i \rightarrow U_i^{**}$, $u \mapsto \beta_u$, w. $\beta_u(\alpha) = \alpha(u)$ — just as for fields. Choose a basis $\xi, ..., \xi_n$ in $N_i$ (over $S$) and complete it to a basis $\xi, ..., \xi_n$ in $U_i^*$ ($n \geq m$). Let

$\xi_1, ..., \xi_n$ be the dual basis in $U_i$ (given by $\alpha_k(\xi_j) = \delta_{kj}$, it exists due to $U_i \rightarrow U_i^{**}$). Then take $\upsilon = \xi_n$.

Step 5: Under the identification $U_i \otimes_{S} U_i^* \rightarrow \text{End}_{S}(U_i)$
all elements from $U_i \otimes N_i$ annihilate $v : (u \otimes 2)(v) = a(u)u = 0$.

By Step 3, so do the elements of $\text{im} \varphi$. Contradiction

$\varphi(1) = 1$. $\square$

2.3) Application: classification of (semi) simple algebras.

The following summarizes two theorems from Lec 19

Theorem: 1) Every finite dimensional semisimple algebra $A$ is isomorphic to $\bigoplus_{i=1}^k \text{Mat}_{n_i}(S_i)$, where $S_i$ is a finite dimensional $F$-algebra.

2) Every finite dimensional simple algebra $A$ is isomorphic to $\text{Mat}_n(S)$ for uniquely determined $n \& S$.

Proof: 1) Let $U_1, \ldots, U_k$ be all pairwise non-isomorphic irreducible $A$-modules. $S_i := \text{End}_A(U_i)^{op}$, $\varphi_i : A \rightarrow \text{End}_F(U_i)$. Let $\varphi = (\varphi_1, \ldots, \varphi_k) : A \rightarrow \bigoplus_{i=1}^k \text{End}_F(U_i)$. By Density theorem, $\text{im} \varphi = \bigoplus_{i=1}^k \text{End}_{S_i}(U_i)$. Choosing an $S_i$-basis in each $U_i$, we identify $\text{End}_{S_i}(U_i) \cong \text{Mat}_{n_i}(S_i)$ w. $n_i = \dim_{S_i} U_i$. So, it
remains to show that \( q \) is injective. Indeed, any element \( a \in k \) acts by \( 0 \) on every direct sum of irreducibles, hence, in particular, on \( A \). But \( a1 = a \Rightarrow a = 0 \).

2): By 1) of Theorem in Sec 1.1 ofLEC 20, we know that \( A \) is semisimple w. unique irreducible module \( U \). So, by 1) (\& its proof), \( A = \text{End}_S(U) = \text{Mat}_n(S) \) for \( n = \dim_k U = \dim_k U / \dim_k S \).

Let’s show that \( A \cong \text{Mat}_n(\tilde{S}) \) for some skew-field (\& finite dimensional \( F \)-algebra \( S \)) \( \Rightarrow \tilde{S} \cong \text{End}_A(U)^{op} \) for the unique irreducible module \( U \). Indeed, we get \( U \cong \tilde{S}^n \). We need to show that any \( \text{Mat}_n(\tilde{S}) \)-linear map \( \tilde{S}^n \rightarrow \tilde{S}^n \) is given as the right multiplication by a unique element of \( \tilde{S} \). This is left as an exercise (hint: prove that \( y \begin{pmatrix} \tilde{x}^1 & \cdots & \tilde{x}^n \end{pmatrix} = \begin{pmatrix} \tilde{y}^1 & \cdots & \tilde{y}^n \end{pmatrix} \) for a unique \( \tilde{y}^i \in \tilde{S} \), then show that \( yv = \tilde{y}^i v \neq v \in \tilde{S}^n \)). \( \square \)

Remark: There’s also a uniqueness statement in 1): the collection \( (S_i, n_i)_{i=1}^k \) is defined uniquely up to a permutation.
2.4) Bonus: Wedderburn–Artin theorem

One can generalize Theorem in Sec 2.3 from algebras to more general rings. Namely, by a semisimple ring we mean an associative ring $A$ whose regular module is a finite direct sum of irreducible modules. Then every finitely generated $A$-module is completely reducible.

The Wedderburn–Artin theorem states that

$$A \cong \bigoplus_{i=1}^{k} \text{Mat}_{n_i}(S_i)$$

for some skew-fields $S_i$. 