# Lecture 22: Finite dimensional associative algebras, IV.

1) Description of submodules.

2) Proof of Density Theorem.

Ref: [E], Secs 3.1, 3.2.

## 1) Description of submodules.

### 1.1) Main result.

Let $\mathbb{F}$ be a field, $A$ be an associative $\mathbb{F}$-algebra. Let $U_1,\dots,U_k$ be pairwise non-isomorphic finite dimensional irreducible $A$-modules, $S_i := \mathrm{End}_A(U_i)^{opp}$ (so that $U_i$ is also a right $S_i$-module and the $A$- and $S_i$-actions on $U_i$ commute). Let $M_i$, $i=1,\dots,k$, be a finite dimensional left $S_i$-module. So $\bigoplus_{i=1}^k U_i \otimes_{S_i} M_i$ becomes a left $A$-module (w. $A$ acting on $U_i \otimes_{S_i} M_i$ via $a(u\otimes m) = au\otimes m$.

We start by constructing a family of $A$-submodules of $\bigoplus_{i=1}^k U_i\otimes_{S_i} M_i$. Let $N_i$ be an $S_i$-submodule of $M_i$. By Sec 1.2 of Lec 21, we can view $U_i\otimes_{S_i} N_i$ as an $\mathbb{F}$-subspace of $U_i\otimes_{S_i} M_i$. From the construction of the $A$-action on $U_i\otimes_{S_i} M_i$

we see that $U_i \otimes_{S_i} N_i$ is an $A$-submodule in $U_i \otimes_{S_i} M_i$. So $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i$ is an $A$-submodule of $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i$.

**Proposition:** Every $A$-submodule $V' \subset V := \bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i$ has the form $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i$ for some $S_i$-submodules $N_i \subset M_i$.

Proof: Let's discuss the idea first. We know that we have natural isomorphisms (Sec 2 of Lec 21)

$$\varphi: \bigoplus_{i=1}^{k} U_i \otimes_{S_i} \mathrm{Hom}_A(U_i, V) \xrightarrow{\sim} V$$

$$\varphi': \bigoplus_{i=1}^{k} U_i \otimes_{S_i} \mathrm{Hom}_A(U_i, V') \xrightarrow{\sim} V'.$$

Next, $\mathrm{Hom}_A(U_i, V')$ embeds as an $S_i$-submodule into $\mathrm{Hom}_A(U_i, V)$. The proof is then checking the details, and, in particular matching various identifications.

Step 1: We claim that $M_i$ is identified w. $\mathrm{Hom}_A(U_i, V)$. Namely, $\mathrm{Hom}_A(U_i, V) = [\mathrm{Hom}_A(U_i, U_j) = \{0\}$ for $i \neq j] =$ $\mathrm{Hom}_A(U_i, U_i \otimes_{S_i} M_i)$. For $m \in M_i$ consider $\varphi_m : U_i \to U_i \otimes_{S_i} M_i$, $\varphi_m(u) = u \otimes m$, an $A$-linear map; $m \mapsto \varphi_m$ is an $S_i$-linear map

$M_i \to \mathrm{Hom}_A(U_i, U_i\otimes_{S_i} M_i)$, where $S_i = \mathrm{End}_A(U_i)^{opp}$ acts on the target by taking compositions on the right, & it's injective (these are left as exercise, for the injectivity use a description of a basis in $U_i \otimes_{S_i} M_i$, Sec 1.2 in Lec 21). Now note that if $\dim_{S_i} M_i = n$, then $U_i\otimes_{S_i} M_i \simeq U_i^{\oplus n}$, so $\dim_{\mathbb{F}} \mathrm{Hom}_A(U_i, U_i\otimes_{S_i} M_i) =$
$n \dim_{\mathbb{F}} \mathrm{Hom}_A(U_i, U_i) = n\dim S_i = \dim_{\mathbb{F}} M_i$.
So $m \mapsto \varphi_m : M_i \xrightarrow{\sim} \mathrm{Hom}_A(U_i, V)$.

Step 2: Recall, Sec 2 of Lec 21, that for an arbitrary finite direct sum $V$ of $U_i$'s, we have

$$\varphi: \bigoplus_{i=1}^{k} U_i \otimes_{S_i} \mathrm{Hom}_A(U_i,V) \xrightarrow{\sim} V,\quad \sum_{i=1}^{k} u_i\otimes\varphi_i \mapsto \sum_{i=1}^{k}\varphi_i(u_i)$$

Thx to Step 1, for $V = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i$ we also have

$$\xi: V \xrightarrow{\sim} \bigoplus_{i=1}^{k} U_i\otimes_{S_i} \mathrm{Hom}_A(U_i,V),\quad \sum_{i=1}^{k} u_i\otimes m_i \mapsto \sum_{i=1}^{k} u_i\otimes \varphi_{m_i}.$$

Since $\varphi_{m_i}(u) = u\otimes m_i$, we have $\varphi\xi = id_V \Rightarrow \xi = \varphi^{-1}$.

Step 3: Consider $N_i := \mathrm{Hom}_A(U_i, V') = \{\varphi \in \mathrm{Hom}_A(U_i,V) \mid \mathrm{im}\,\varphi \subset V'\}$
This is a subspace in $M_i = \mathrm{Hom}_A(U_i, V)$. Moreover, it's an $S_i$-submodule: $S_i = \mathrm{End}_A(U_i)^{opp}$ acts on $\mathrm{Hom}_A(U_i,V)$ by $s\varphi = \varphi\circ s$, so

fixes $\mathrm{Hom}_A(U_i, V')$. We claim that $V' = \bigoplus_{i=1}^{k} U_i \otimes_S N_i$, which will finish the proof. Indeed, let $\psi' : \bigoplus_{i=1}^{k} U_i \otimes_S N_i \longrightarrow V'$ be the analog of $\psi$ for $V'$. The following diagram commutes by the construction of $\psi$ (& $\psi'$):

$$\begin{tikzcd}
\bigoplus_{i=1}^{k} U_i \otimes_S N_i \arrow[r, "\psi'", "\sim"'] \arrow[d, hook] & V' \arrow[d, hook] \\
\bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i \arrow[r, "\psi", "\sim"'] & V
\end{tikzcd}$$

The vertical maps are inclusions, and we identify $M_i$ w. $\mathrm{Hom}_A(U_i, V)$ as in Step 1. In particular, we see that $V'$ is indeed of the form $\bigoplus_{i=1}^{k} U_i \otimes_S N_i$ for $S_i$-submodules $N_i \subset M_i$ $\square$

Exercise: Show that $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i'$ (for submodules $N_i, N_i' \subset M_i$) implies $N_i = N_i'$ $\forall i$ (hint: use bases in tensor products introduced in Sec 1.2 of Lec 21). This shows that $N_i$'s in Proposition are uniquely recovered from $V'$.

Remark: Thx to Sec 2 of Lec 21, Proposition describes submodules in an arbitrary completely reducible module.

## 2) Proof of Density Theorem.

### 2.1) Statement

The theorem was stated in Sec 3 of Lec 20.

**Theorem:** Let $A$ be an associative algebra and $U_1, \dots U_k$ be its pairwise nonisomorphic finite dimensional irreducibles. Let $S_i := \operatorname{End}_A(U_i)^{opp}$ and let $\varphi_i : A \to \operatorname{End}_{\mathbb{F}}(U_i)$ be the homomorphism corresponding to the $A$-module $U_i$. Then the image of $(\varphi_1, \dots \varphi_k)$ is $\bigoplus_{i=1}^{k} \operatorname{End}_{S_i}(U_i)$.

Before we get to the proof, let's record

**Corollary/Exercise:** Suppose that $\dim_{\mathbb{F}} A < \infty$. Then the number of irreducible (automatically, finite dimensional) $A$-modules (up to iso) is $\leqslant \dim A$.

### 2.2) Proof of Density Theorem

Again, let's start w. an idea. We'll identify $\bigoplus_{i=1}^{k} \operatorname{End}_{S_i}(U_i)$

w. $\bigoplus_{i=1}^{k} U_i \otimes_{S_i} U_i^*$ and then use Proposition to show that $\operatorname{im}\varphi = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i$ for $N_i \subset U_i^*$. If $N_i \subsetneq U_i^*$ for some $i$, then, as we'll check, every element of $\operatorname{im}\varphi$ annihilates a nonzero vector in $U_i$. This will give a contradiction.

Step 1: Analogously to Sec 1.3 of Lec 21, for a right $S$-module $U$ (w. $\dim_S U < \infty$) we have an isomorphism

$$(*) \qquad U \otimes_S U^* \xrightarrow{\sim} \operatorname{End}_S(U).$$

Here $U^* := \operatorname{Hom}_S(U, S)$, we identify $U$ w. $\operatorname{Hom}_S(S, U)$ via $u \mapsto [s \mapsto us]$. Then $(*)$ is given by

$$\psi \otimes \alpha \mapsto \psi \circ \alpha, \quad \psi \in \operatorname{Hom}_S(S, U), \alpha \in \operatorname{Hom}_S(U, S).$$

Step 2: Suppose now $U$ is a left $A$-module in such a way that the actions of $A$ & $S$ commute. So $U \otimes_S U^*$ acquires a left $A$-module structure via $a(u \otimes \alpha) = au \otimes \alpha$, $\operatorname{End}_S(U)$ is also a left $A$-module: via

$$a\xi = a_U \circ \xi, \quad a \in A, \xi \in \operatorname{End}_S(U).$$

It's left as an exercise to check that $(*)$ is an $A$-

module homomorphism.

Step 3: So, as an $A$-module,

$$\bigoplus_{i=1}^{k} \mathrm{End}_{S_i}(U_i) = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i, \quad M_i := U_i^*.$$

Note that $\mathrm{im}\,\varphi \subset \bigoplus_{i=1}^{k} \mathrm{End}_{S_i}(U_i)$ is an $A$-submodule. By Proposition in Sec 1, $\exists\ N_i \subset M_i = U_i^*$ s.t.

$$\mathrm{im}\,\varphi = \bigoplus_{i=1}^{k} U_i \otimes_{S_i} N_i.$$

We need to show that $N_i = M_i\ \forall i$. Assume the contrary: $N_i \subsetneq M_i$.

Step 4: We claim $\exists\ v \in U_i$ s.t. $\alpha(v) = 0\ \forall\ \alpha \in N_i$. For this, observe first that $U_i \xrightarrow{\sim} U_i^{**}$, $u \mapsto \beta_u$, w. $\beta_u(\alpha) = \alpha(u)$ – just as for fields. Choose a basis $\alpha_1, \dots \alpha_m$ in $N_i$ (over $S$) and complete it to a basis $\alpha_1, \dots \alpha_n$ in $U_i^*$ $(n > m)$. Let $u_1, \dots u_n$ be the dual basis in $U_i$ (given by $\alpha_k(u_j) = \delta_{kj}$, it exists thx to $U_i \xrightarrow{\sim} U_i^{**}$). Then take $v := u_n$.

Step 5: Under the identification $U_i \otimes_S U_i^* \xrightarrow{\sim} \mathrm{End}_S(U_i)$

all elements from $U_i \otimes N_i$ annihilate $v$: $[u\otimes\alpha](v) = \alpha(v)u = 0$. By Step 3, so do the elements of $\operatorname{im}\varphi$. Contradiction b/c $\varphi(1)=1$. $\square$

## 2.3) Application: classification of (semi)simple algebras.

The following summarizes two theorems from Lec 19.

**Theorem:** 1) Every finite dimensional semisimple algebra $A$ is isomorphic to $\bigoplus_{i=1}^{k} \operatorname{Mat}_{n_i}(S_i)$, where $S_i$ is a finite dimensional $\mathbb{F}$-algebra.

2) Every finite dimensional simple algebra $A$ is isomorphic to $\operatorname{Mat}_n(S)$ for uniquely determined $n$ & $S$.

Proof: 1): Let $U_1, \ldots U_k$ be all pairwise non-isomorphic irreducible $A$-modules, $S_i := \operatorname{End}_A(U_i)^{opp}$, $\varphi_i: A \to \operatorname{End}_{\mathbb{F}}(U_i)$ & $\varphi = (\varphi_1, \ldots \varphi_k): A \longrightarrow \bigoplus_{i=1}^{k} \operatorname{End}_{\mathbb{F}}(U_i)$. By Density theorem, $\operatorname{im}\varphi = \bigoplus_{i=1}^{k} \operatorname{End}_{S_i}(U_i)$. Choosing an $S_i$-basis in each $U_i$, we identify $\operatorname{End}_{S_i}(U_i) \xrightarrow{\sim} \operatorname{Mat}_{n_i}(S_i)$ w. $n_i := \dim_{S_i} U_i$. So, it remains to show that $\varphi$ is injective. Indeed, any element $a \in \ker\varphi$ acts by $0$ on every direct sum of irreducibles, hence, in particular, on $A$. But $a1 = a \Rightarrow a = 0$.

2): By 1) of Theorem in Sec 1.1 of Lec 20, we know that $A$ is semisimple w. unique irreducible module $U$. So, by 1) (& its proof), $A = \mathrm{End}_S(U) = \mathrm{Mat}_n(S)$ for $n = \dim_S U = \dim_{\mathbb{F}} U / \dim_{\mathbb{F}} S$.

Let's show that $A \simeq \mathrm{Mat}_n(\tilde{S})$ for some skew-field (& finite dimensional $\mathbb{F}$-algebra $S$) $\Rightarrow \tilde{S} \simeq \mathrm{End}_A(U)^{opp}$ for the unique irreducible module $U$. Indeed, we get $U \simeq \tilde{S}^n$. We need to show that any $\mathrm{Mat}_n(\tilde{S})$-linear map $\tilde{S}^n \xrightarrow{\eta} \tilde{S}^n$ is given as the right multiplication by a unique element of $\tilde{S}$. This is left as an exercise (hint: prove that $\eta\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}s_\eta\\0\\\vdots\\0\end{pmatrix}$ for a unique $s_\eta \in \tilde{S}$, then show that $\eta v = s_\eta v \;\forall\, v \in \tilde{S}^n$). $\square$

Remark: There's also a uniqueness statement in 1): the collection $(S_i, n_i)_{i=1}^k$ is defined uniquely up to a permutation.

## 2.4) Bonus: Wedderburn-Artin theorem

One can generalize Theorem in Sec 2.3 from algebras to more general rings. Namely, by a semisimple ring we mean an associative ring $A$ whose regular module is a finite direct sum of irreducible modules. Then every finitely generated $A$-module is completely reducible.

The Wedderburn-Artin theorem states that

$$A \simeq \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(S_i)$$

for some skew-fields $S_i$.