Lecture 23: Finite dimensional associative algebras, V.

1) Criteria for semisimplicity.

2) Applications to representations of finite groups.

3) Bonus: analogous results for Lie algebras

Refs: [V], Sec 11.3.

1.0) Recap

Let $\mathbb{F}$ be a field and A be a finite dimensional $\mathbb{F}$-algebra. We have seen that if A is semisimple, then

$$A \simeq \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(S_i)$$

where S_i are skew-fields that are finite dimensional $\mathbb{F}$-algebras (Thm in Sec 2.3 of Lec 22), and that the irreducible A-modules are exactly $S_i^{n_i}$ $(i=1,..k)$ – 3) of Thm in Sec 1.1 of Lec 20. We note that if $\mathbb{F}$ is algebraically closed, then the situation simplifies as $S_i = \mathbb{F}$ $\forall i$.

The goal of this section is to state two criteria of semisimplicity: via "nilpotent ideals" and via the trace form.

1.1) Nilpotent ideals.

Let $I, J \subset A$ be two-sided ideals. We can define their product $IJ = \mathrm{Span}_{\mathbb{F}}(ab \mid a \in I, b \in J)$, this is also a two-sided ideal (exercise). The product of ideals is associative, so, for $n \in \mathbb{Z}_{\geqslant 0}$, it makes sense to speak about the power I^n.

Definition: A two-sided ideal $I \subset A$ is called *nilpotent* if $I^n = \{0\}$ for some $n > 0$.

Example: Let A be the subalgebra of all upper-triangular matrices in $\mathrm{Mat}_m(\mathbb{F})$ & $I \subset A$ is the subspace of all strictly upper triangular matrices. Then I is a two-sided ideal & it's nilpotent ($I^m = \{0\}$).

Lemma: For a two-sided ideal $I \subset A$, TFAE:

a) I is nilpotent.

b) I acts by 0 on every irreducible A-module.

Proof: a) $\Rightarrow$ b). For a two-sided ideal $J \subset A$ & an A-module M define $JM = \mathrm{Span}_{\mathbb{F}}(bm \mid b \in J, m \in M)$, this is an A-submodule of M (exercise). And note that for two ideals J_1, J_2 we have $(J_1 J_2)M = J_1(J_2 M)$.

Assume U is an irreducible A-module s.t. $IU \neq \{0\}$. Since $IU \subset U$ is a submodule, $IU = U$. But then $I^n U = U$ $\forall n$. Contradiction w $I^n = \{0\}$ for some n.

b) $\Rightarrow$ a): Choose a "Jordan-Hölder" filtration on A: $\{0\} = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = A$, where M_i's are A-submodules & M_i/M_{i-1} is irreducible. Then $I(M_i/M_{i-1}) = \{0\} \Leftrightarrow IM_i \subset M_{i-1}$ $\forall i$. So $I^n A = \{0\} \Rightarrow I^n = \{0\}$. $\square$

Corollary: 1) A has the unique maximal (w.r.t. $\subset$) nilpotent ideal (called the radical of A and denoted by $\mathrm{Rad}(A)$).

2) TFAE:

(a) A is semisimple

(b) $\mathrm{Rad}(A) = \{0\}$.

Proof: 1) Recall (Corollary in Sec 2.1 of Lec 22) that A has only finitely many irreducibles, say $U_1, \dots U_k$. Let $\varphi_i: A \rightarrow \mathrm{End}_{\mathbb{F}}(U_i)$ be the corresponding homomorphism. Thx (a)$\Leftrightarrow$(b). $\mathrm{Rad}(A) := \bigcap_{i=1}^{k} \ker \varphi_i$ is the unique maximal nilpotent ideal.

2) (a) $\Rightarrow$ (b): $\mathrm{Rad}(A)$ acts by 0 on every completely reducible (fin. dim.) A-module. And A is completely reducible as an A-module $\Rightarrow \mathrm{Rad}(A) = \{0\}$.

(b) $\Rightarrow$ (a): In the notation of proof of 1), set

$$\varphi = (\varphi_1, \dots \varphi_k): A \longrightarrow \bigoplus_{i=1}^{k} \mathrm{End}_{\mathbb{F}}(U_i)$$

$\mathrm{Rad}(A) = \{0\} \overset{1)}{\Longleftrightarrow} \varphi$ is injective. Then A is semisimple by Proposition in Sec 1.1 of Lec 20. $\square$

Exercise : Let A be the subalgebra of upper triangular matrices in $\mathrm{Mat}_n(\mathbb{F})$. Then $\mathrm{Rad}(A)$ is the two-sided ideal of all strictly upper triangular matrices.

1.2) Trace form.

Let A be a finite dimensional $\mathbb{F}$-algebra and V be a finite dimensional A-module. Consider the bilinear form on A given by $(a,b)_V := tr(a_V b_V)$. It's symmetric.

Exercise: if $U \subset V$ is a submodule, then
$$(a,b)_V = (a,b)_U + (a,b)_{V/U} \quad \forall\, a,b \in A.$$

We will be primarily interested in the case of $V=A$, here we just write $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_A$. The property we care about is being non-degenerate.

Examples: 1) Let G be a finite group & $A = \mathbb{F}G$. Then $(a,b) = \chi_{\mathbb{F}G}(ab)$. So for $a,b \in G (\subset \mathbb{F}G)$ we have $(a,b) = |G|\,\delta_{ab,e}$. Hence $(\cdot,\cdot)$ is nondegenerate if $\operatorname{char}\mathbb{F} \nmid |G|$ (and is 0 else): this is because we can find dual bases: the basis $\frac{1}{|G|}g^{-1}, g \in G$, is dual to the basis $g \in G$: $(\frac{1}{|G|}g^{-1}, h) = \delta_{gh}\ \forall g,h \in G$.

2) Let $A = \mathrm{Mat}_n(\mathbb{F})$. Recall that we have an isomorphism of A-modules $A \simeq (\mathbb{F}^n)^{\oplus n}$. So $(a,b) = n(a,b)_{\mathbb{F}^n} = n\,\mathrm{tr}(ab)$. Note that $(a,b)_{\mathbb{F}^n}$ is nondegenerate b/c $\mathrm{tr}(E_{ij}E_{ji}) = \mathrm{tr}(E_{ii}) = 1$. $\forall i,j$. So $(\cdot,\cdot)$ is nondegenerate if $\mathrm{char}\,\mathbb{F} \nmid n$ (and is 0 else)

3) Let $A = S$ be a skew-field. Suppose first $\mathrm{char}\,\mathbb{F} \nmid \dim_{\mathbb{F}} S$. Then $(\cdot,\cdot)$ is non-degenerate: $(s, s^{-1}) = \mathrm{tr}((ss^{-1})_S) = \mathrm{tr}(1_S) = \dim_{\mathbb{F}} S \neq 0$ $\forall s \in S$. In fact, if $\mathrm{char}\,\mathbb{F} \mid \dim_{\mathbb{F}} S$, then $(\cdot,\cdot) = 0$, one can see this using "base change to the algebraic closure."

3') Let $A = \mathrm{Mat}_n(S)$, where S is a skew-field (and a finite dimensional $\mathbb{F}$-algebra). As above, $(a,b) = n(a,b)_{S^n}$. The form $(a,b)_{S^n}$ is non-degenerate if $\mathrm{char}\,\mathbb{F} \nmid \dim_{\mathbb{F}} S$. Indeed, choose a basis $s_1, \dots, s_m \in S$ and it's dual basis $s_1', \dots s_m' \in S$ w.r.t. $(\cdot,\cdot)$. Similarly to Example 2), we have the following dual $\mathbb{F}$-bases in $\mathrm{Mat}_n(\mathbb{F})$: $s_\ell E_{ij}$ $(\ell = 1,\dots m,\ i,j = 1,\dots n)$ & $s_\ell' E_{ji}$ (exercise). Our conclusion is that if $\mathrm{char}\,\mathbb{F} \nmid n \dim_{\mathbb{F}} S$, then $(\cdot,\cdot)$ is non-degenerate.

Here's how the non-degeneracy of $(\cdot,\cdot)$ is relevant to the semisimplicity.

Theorem: 1) If $(\cdot,\cdot)$ is non-degenerate, then A is semisimple.

2) If $\operatorname{char}\mathbb{F}=0$, then the converse is true.

3) If $\mathbb{F}$ is algebraically closed & $\operatorname{char}\mathbb{F}=p$, TFAE

(a) $(\cdot,\cdot)$ is non-degenerate.

(b) A is semisimple & all irreducibles have dimensions coprime to p.

Proof: 1) Thx to Corollary in Sec 1.1, it's enough to show that any nilpotent ideal I is 0. Let $a\in A$, $b\in I \Rightarrow ab\in I \Rightarrow$ $(ab)_A$ is nilpotent $\Rightarrow (a,b)=\operatorname{tr}((ab)_A)=0$. Since $(\cdot,\cdot)$ is non-degenerate, $b=0$.

2) We use the classification, $A=\bigoplus_{i=1}^{k} \operatorname{Mat}_{n_i}(S_i)$. We remark that if A is presented as $\bigoplus_{i=1}^{k} A_i$, then A_i is orthogonal to A_j w.r.t. $(\cdot,\cdot)$ b/c $a_i a_j=0$ for $a_i\in A_i, a_j\in A_j$ $(i\neq j)$ & $(\cdot,\cdot)|_{A_i}$ coincides w. the trace form for A_i. The forms for $A_i=\operatorname{Mat}_{n_i}(S_i)$ are nondegenerate (Ex. 3) so $(\cdot,\cdot)$ is nondegenerate for A as well.

3) Left as an exercise. $\square$

Exercise*: if $\operatorname{char}\mathbb{F}=0$, then $\operatorname{Rad}(A)=A^{\perp}$.

Remark: The condition that $\mathbb{F}$ is algebraically closed in 3) can be omitted (compare to Example 3).

2) Applications to representations of finite groups.

An immediate application is as follows (there are more, see e.g. Problems 2 & 3 in HW5).

Theorem: Let G be a finite group. Suppose $\operatorname{char}\mathbb{F} \nmid |G|$.

1) Finite dimensional representations of G are completely reducible.

2) If $\mathbb{F}$ is algebraically closed, then the dimensions of irreducibles are not divisible by $\operatorname{char}\mathbb{F}$.

3) If $\mathbb{F}$ is algebraically closed, then the number of irreducibles equals to the number of conjugacy classes in G.

Proof: By Example 1 in Sec 1.2, $(\cdot,\cdot)$ is non-degenerate on $\mathbb{F}G$. Now 1) follows from 1) of Theorem in Sec 1.2. 2) is vacuous if char $\mathbb{F}=0$ and follows from 3) of the previous theorem if char $\mathbb{F}=p$.

We proceed to 3). Recall that to an associative algebra A we can assign its center $Z(A)=\{z\in A \mid za=az \ \forall a\in A\}$. For $A=\mathbb{F}G$, we have $Z(\mathbb{F}G)=\{\sum_{g\in G} a_g g \mid a_g$ is constant on conjugacy classes$\}$ $\Rightarrow \dim Z(\mathbb{F}G)=\#$ conj. classes in G, see Sec 1 of Lec 8. On the other hand, by 1) and the classification of semisimple algebras, $\mathbb{F}G\simeq \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(\mathbb{F})$ for some n_i. Now, 3) follows from $\dim Z(\bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(\mathbb{F}))=k$, which is a consequence of the next exercise. $\square$

Exercise: 1) Let S be a skew-field and $n>0$. Then

$$Z(\mathrm{Mat}_n(S))=\{\mathrm{diag}(z,z,\ldots z) \mid z\in Z(S)\}$$

(hint: commute an element in the center w. E_{ij} & sE_{ii}, $s\in S$).

2) $Z(\bigoplus_{i=1}^{k} A_i)=\bigoplus_{i=1}^{k} Z(A_i)$.

3) In particular, $\dim Z(\bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(\mathbb{F}))=k$.

Remark: We can remove the condition that $\mathbb{F}$ is algebraically closed in 2) of the theorem. If we remove this assumption in 3), we get that

$$\# \text{ of irreducibles} = k \leq \sum_{i=1}^{k} \dim_{\mathbb{F}} Z(S_i) = \# \text{ of conj. classes.}$$

3) Bonus: analogous results for Lie algebras

Let $\mathbb{F}$ be a characteristic 0 field. Let's summarize the four equivalent definitions of a semisimple finite dimensional associative algebras, A. TFAE:

(i) A is isomorphic to a direct sum of simple algebras.

(ii) All finite dimensional representations of A are completely reducible

(iii) $Rad(A) = \{0\}$

(iv) $(\cdot,\cdot)$ is nondegenerate.

It turns out that this carries (w. suitable modifications) to the more interesting setting of Lie algebras (see Bonus for Lec 3).

(i): we need the notion of an ideal in a Lie algebra $\mathfrak{g}$. This is a subspace $\mathfrak{a} \subset \mathfrak{g}$ s.t. $x \in \mathfrak{g}, y \in \mathfrak{a} \Rightarrow [x,y] \in \mathfrak{a}$. Equivalently, this is the kernel of a homomorphism from $\mathfrak{g}$. By a simple Lie algebra we mean a Lie algebra $\mathfrak{g}$ of $\dim > 1$ w/o proper two-sided ideals (if $\dim \mathfrak{g} = 1$, the bracket is 0, we exclude this case for the same reason as for excluding the cyclic groups in our definition of simple groups).

(ii): generalizes in a straightforward way.

(iii): We need to redefine $Rad(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$. Note that an ideal is also a Lie subalgebra (unlike in the associative algebra case, where we require that subalgebras contain 1). For a Lie algebra $\mathfrak{h}$ define its subalgebra (in fact, an ideal) $\mathfrak{h}^{(1)} = Span_{\mathbb{F}}([x,y] \mid x,y \in \mathfrak{h})$. Then inductively define $\mathfrak{h}^{(i)} := (\mathfrak{h}^{(i-1)})^{(1)}$ for $i > 1$. We say that $\mathfrak{h}$ is solvable if $\mathfrak{h}^{(i)} = \{0\}$ for some i. For example, the Lie algebra of upper triangular matrices in $Mat_n(\mathbb{F})$ is solvable.

One can show that every finite dimensional Lie algebra

$\mathfrak{g}$ contains a unique maximal (w.r.t. $\subset$) solvable ideal. This ideal is called the radical of $\mathfrak{g}$.

(iv) An analog of the regular representation for $\mathfrak{g}$ is the adjoint representation in $\mathfrak{g}$: $ad(x).y := [x,y]$. With this we can define the Killing form (Killing was a German mathematician) by $(x,y) := tr(ad(x)ad(y))$.

Theorem: Let $\mathfrak{g}$ be a finite dimensional Lie algebra. TFAE:

i) $\mathfrak{g}$ is isomorphic to the direct sum of simple Lie algebras.

ii) Every finite dimensional representation of $\mathfrak{g}$ is completely reducible.

iii) $Rad(\mathfrak{g}) = \{0\}$.

iv) The Killing form on $\mathfrak{g}$ is non-degenerate.

The Lie algebras satisfying (i) are called semisimple. The study of (semi)simple Lie algebras (their classification and also the classification of their finite dimensional irreducible representations) over an algebraically closed char 0 is the central part of a 1st class on Lie groups/algebras.