Lecture 3: Basics, II. Amended 1/25

1) Homomorphisms of representations.
2) Associative algebras and their representations.
3*) Bonus: Lie algebras & universal enveloping algebras.

1) Homomorphisms of representations.

For all algebraic structures, there's the notion of a homomorphism. Here's what it looks like for representations.

Let G be a group, $\mathbb{F}$ be a field, and U, V be representations of G over $\mathbb{F}$.

Definition: An $\mathbb{F}$-linear map $\varphi: U \to V$ is a *homomorphism* (of representations of G) if $g.\varphi(u) = \varphi(g.u)$ $\forall g \in G, u \in U$.

Examples: 0) The zero map $U \to V$ is a homomorphism.

1) The identity map $Id_V: V \to V$ is a homomorphism.

2) Let $U \subset V$ be a subrepresentation. The inclusion map $U \to V$, $u \mapsto u$, and the projection map $V \to V/U$, $v \mapsto v+U$,

are homomorphisms: the representations in U & V/U are constructed to make this the case (Secs 2.2 & 2.3 in Lec 2).

Exercise: Let X, Y be sets w. G-actions and $\varphi: X \to Y$ be a map intertwining the G-actions: $\varphi(g.x) = g.\varphi(x)$ $\forall g \in G$, $x \in X$. Consider the map $\mathrm{Fun}(Y, \mathbb{F}) \longrightarrow \mathrm{Fun}(X, \mathbb{F})$, $f \mapsto f \circ \varphi$. Show that it is a homomorphism of representations.

Let's proceed to discussing properties of homomorphisms. They mirror properties of linear maps and their proofs are left as exercises.

Lemma 1: Let $\varphi: U \to V$ be a homomorphism of representations. Then $\ker \varphi \subset U$, $\operatorname{im} \varphi \subset V$ are subrepresentations.

Lemma 2: Let U, V, W be representations of G.

a) If $\varphi: U \to V$, $\varphi': V \to W$ are homomorphisms, then so is $\varphi' \circ \varphi: U \to W$.

b) If $\varphi: U \to V$ is a bijective homomorphism, then $\varphi^{-1}: V \to U$ is also a homomorphism. In this case we say that φ is an isomorphism.

c) If $\varphi, \varphi': U \to V$ are homomorphisms, then so are $\varphi + \varphi'$, $a\varphi: U \to V$ for $a \in \mathbb{F}$.

Remarks: 1) If there is an isomorphism between representations U & V, then we say that they are isomorphic. This means that they behave in the same way, and one usually seeks to understand representations up to isomorphism.

2) Let $\mathrm{Hom}(U,V)$ denote the space of linear maps $U \to V$. Part c) tells us that homomorphisms of representations form a subspace in $\mathrm{Hom}(U,V)$, to be denoted by $\mathrm{Hom}_G(U,V)$.

2) Associative algebras and their representations.

Associative algebras is another class of algebraic structures whose representations we want to study. In a way, this is the most important class: for all other algebraic structures (groups & Lie algebras being the most important examples), their representations are, in fact, representations of suitable associative algebras.

2.1) Bilinear maps.

Definition: Let U, V, W be vector spaces. A map $\beta: U\times V \to W$ is bilinear if it's linear in each of the arguments if we fix the other (e.g. $\forall\, u\in U$, the map $v \mapsto \beta(u,v): V\to W$ is linear, and the same for each fixed v).

Example: Let X be a vector space and $End(X)$ be the space of linear operators $X\to X$. Then the composition map $End(X)\times End(X) \longrightarrow End(X)$, $(\varphi,\psi)\mapsto \varphi\circ\psi$, is bilinear.

Rem: Similarly to linear maps, bilinear ones a completely determined by the images of basis elements: if $u_1, \dots u_m \in U$, $v_1,\dots,v_n\in V$ are bases, then

$$\beta\Big(\sum_i a_i u_i, \sum_j b_j v_j\Big) = \sum_{i,j} a_i b_j\, \beta(u_i, v_j) \quad a_i, b_j \in \mathbb{F}.$$

2.2) Algebras

Definition: An **algebra** over $\mathbb{F}$ is an $\mathbb{F}$-vector space equipped with a bilinear map $A \times A \to A$, the **product**.

Any algebra is a ring, so we can talk about associative, commutative and unital algebras. In this course, we are interested in unital associative algebras (and all associative algebras we consider are unital, so we omit that adjective). An example of an associative algebra is $End(X)$ for a vector space X, the product is the composition, see Example in Sec 2.1. Here's another important example.

Example (the group algebra) Let G be a group (finite, for simplicity). Define its **group algebra**, to be denoted by $\mathbb{F}G$, as a vector space w. basis v_g, $g \in G$, and the unique product given on the basis by $v_g v_h := v_{gh}$. Note that, by this definition, G can be viewed as a subset of $\mathbb{F}G$ via $g \mapsto v_g$ (and we write g instead of v_g).

Lemma: The algebra $\mathbb{F}G$ is associative w. unit $e \in G (\subset \mathbb{F}G)$. It's commutative iff G is abelian.

Proof: We'll check the associativity, everything else is an exercise. Let $a = \sum_{g\in G} a_g g$, $b = \sum_{h\in G} b_h h$, $c = \sum_{k\in G} c_k k$ $(a_g, b_h, c_k \in \mathbb{F})$
Then $(ab)c = \sum_{g,h,k} a_g b_h c_k (gh)k = [(gh)k = g(hk)] =$
$\sum_{g,h,k} a_g b_h c_k g(hk) = a(bc)$ $\square$

2.3) Representations of associative algebras

Definition: Let A be an associative (and unital) algebra & V is an $\mathbb{F}$-vector space. By a representation of A in V we mean an associative algebra homomorphism $A \to \mathrm{End}(V)$
i.e a linear map η satisfying $\eta(ab) = \eta(a)\eta(b)$ $\forall a, b \in A$
(we make the convention that all associative algebra homomorphisms we consider send 1 to 1).

As in the case of groups, we can equivalently define a representation as a map $A \times V \to V$, $(a,v) \mapsto av$, that is bilinear, associative (i.e. $(ab)v = a(bv)$) and satisfies the unit axiom: $1v = v$ ($\forall\, v \in V$). When we use this description, we usually say that V is an ==A-module==.

Examples: 1) V is an $\mathrm{End}(V)$-module (via the identity homomorphism $\mathrm{End}(V) \longrightarrow \mathrm{End}(V)$).

2) A is a module over itself via $(a,b) \mapsto ab$. This is called the ==regular module== (or representation).

Remark: The general constructions explained for groups in Lecture 2 (direct sums, sub and quotient representations) work for representations of associative algebras. The same is the case for homomorphisms of representations (Sec 1): a homomorphism of A-modules $U \to V$ is the same as an A-linear map.

2.4) Representation of G vs $\mathbb{F}G$.

It turns out that the representations of G are in a natural bijection with those of $\mathbb{F}G$.

Indeed, let $\eta: \mathbb{F}G \to End(V)$ be a representation. Define the map $\rho_\eta: G \to End(V)$ as $\eta|_G$. Then $\rho_\eta(e) = Id_V$ & $\rho_\eta(gh) = \rho_\eta(g)\rho_\eta(h)$ b/c η is a homomorphism of associative algebras. Also $\rho_\eta(g)\rho_\eta(g^{-1}) = \rho_\eta(e) = Id_V$ so $im\, \rho_\eta \subset GL(V)$, hence ρ_η is a group homomorphism $G \to GL(V)$ a.k.a. a group representation.

In the opposite direction, let $\rho: G \to GL(V)$ be a group homomorphism. Define $\eta_\rho: \mathbb{F}G \to End(V)$ by

$$\eta_\rho\Big(\sum_{g\in G} a_g g\Big) := \sum_{g\in G} a_g \rho(g), \quad a_g \in \mathbb{F}.$$

Exercise: η_ρ is an associative algebra homomorphism. Moreover, the maps $\rho \mapsto \eta_\rho$ & $\eta \mapsto \rho_\eta$ are mutually inverse.

This establishes the bijection of interest.

Example: Identify $Fun(G, \mathbb{F})$ (Ex 2 in Lec 2, Sec 1) w. $\mathbb{F}G$ via $\delta_g \mapsto g$. Viewed as an $\mathbb{F}G$-module, $Fun(G,\mathbb{F})$ is the regular module.

Side remark 1: The transfer of the product from $\mathbb{F}G$ to $\mathrm{Fun}(G,\mathbb{F})$ is known as the convolution:

$$f_1 * f_2(g) = \sum_{h\in G} f_1(gh^{-1})f_2(h)$$

There are many settings in Math, where the convolution appears and our current setting is the most elementary.

Remark: Why should we view representations of G as those of $\mathbb{F}G$? A practical reason is that this provides a convenient way to construct representations (e.g. in left ideals $I \subset \mathbb{F}G$ or the quotients $\mathbb{F}G/I$). We'll explore this when we need it. Besides, the setting of associative algebras provides a convenient language to talk about representations of more general algebraic structures.

Side remark 2: The group algebra construction can be interpreted via adjoint functors (studied, e.g. in MATH 380). Namely, there's a "group algebra functor" going from the category of groups to the category of associative

$\mathbb{F}$-algebras that sends G to $\mathbb{F}G$ and does the only natural thing to homomorphisms). It's left adjoint to a "forgetful functor" that sends an associative algebra A to the group of its invertible elements.

3*) Bonus: Lie algebras & universal enveloping algebras.

3.1) Lie algebras.

This is a class of non-associative algebras that is of great importance for several areas of Mathematics, incl. Representation theory.

Definition: An algebra $\mathfrak{g}$ (w. product denoted by $[\cdot,\cdot]$ and called the Lie bracket or the commutator) is a Lie algebra if the product satisfies the following axioms:

(Skew-symmetry): $[x,x]=0 \ \forall\, x\in\mathfrak{g}$

(Jacobi identity): $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0, \ \forall\, x,y,z\in\mathfrak{g}$.

Note the first axiom implies more familiar $[x,y]=-[y,x]$

(and is equivalent to it if char $\mathbb{F} \neq 2$).

Example 1: Let A be an associative algebra. Then it becomes a Lie algebra w. $[a,b] := ab - ba$. This can be viewed as a forgetful functor from the category of associative algebras to the category of Lie algebras. For example, if we apply this to $A := End(V)$ we get the general linear Lie algebra $\mathfrak{gl}(V)$.

Example 2: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. It makes sense to talk about real or complex manifolds. By a Lie group one means a manifold, G, equipped with a group structure so that the product $G \times G \longrightarrow G$ and the inverse map $G \longrightarrow G$ are C^∞. For example, $GL_n(\mathbb{F})$, $SL_n(\mathbb{F})$ are $O_n(\mathbb{F})$ (the orthogonal matrices) are Lie groups (over $\mathbb{F}$), while the group U_n of unitary (complex-valued) matrices is a real Lie group.

For a Lie group G, its tangent space $T_e G$ has a naturel Lie algebra structure (roughly speaking by differentiating the group commutator). For example, for $G = GL_n(\mathbb{F})$, we get $T_e G = \mathfrak{gl}_n(\mathbb{F})$, while for $G = SL_n(\mathbb{F})$, we get $T_e G = \mathfrak{sl}_n(\mathbb{F})$, the subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ consisting of all trace 0 matrices.

The study of many questions about Lie groups, e.g. about their representations can be reduced to Lie algebras ("linearization"). This is the initial reason why mathematicians care about Lie algebras.

3.2) Representations & universal enveloping algebras.

A representation of a Lie algebra $\mathfrak{g}$ in a vector space V is a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ $(= \operatorname{End}(V))$. As in the case of groups, a representation of a Lie algebra $U(\mathfrak{g})$ is the same thing as a representation of a suitable associative algebra called the universal enveloping algebra and denoted by $U(\mathfrak{g})$. This algebra is constructed as the quotient of the tensor algebra $T(\mathfrak{g})$ $(:= \bigoplus_{i=0}^{\infty} \mathfrak{g}^{\otimes i})$ by the two-sided ideal generated by the elements $x \otimes y - y \otimes x - [x,y]$ for $x, y \in \mathfrak{g}$. Note that we have a vector space embedding

$\mathfrak{g} \hookrightarrow T(\mathfrak{g})$ (deg 1 tensors). Its composition with the projection $T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$ is a Lie algebra homomorphism, denote it by ι. Composing with ι defines a map from the set of representations of $U(\mathfrak{g})$ in V to that of representations of $\mathfrak{g}$ in V. It is a bijection, left as an exercise. Even more is true, compare to Side Remark 2 in Sec 2.4: taking the universal enveloping algebra is a (part of a) functor from the category of Lie algebras to that of associative algebras. This functor is left adjoint to the forgetful functor from the category of associative algebras to that of Lie algebras (see Ex 1 in Sec 3.1).