Lecture 4

1) Tensor products and duals of vector spaces.
2) Tensor products and duals of group representations.

Refs: [V], Sec 8.1.

1) Tensor products and duals of vector spaces.

We work w. vector spaces over a field $\mathbb{F}$.

1.1) Tensor products: construction.

In Sec 2.1 of Lec 3 we talked about bilinear maps.

For two vector spaces U, V, their tensor product $U\otimes V$ is another vector space together w. a bilinear map $U\times V\longrightarrow U\otimes V$ that have some "universality property" to be stated below.

We will need only the case when U & V are finite dimensional. Fix bases $u_1,\dots u_m\in U$, $v_1,\dots v_n\in V$. Define $U\otimes V$ as the vector space w. basis of symbols $u_i\otimes v_j$, $i=1,\dots,m$, $j=1,\dots n$ (so of dimension mn) together w. the unique bilinear map, denote it temporarily by β, given on the basis elements by $\beta(u_i,v_j):=u_i\otimes v_j$. For general $u\in U$,

$v \in V$, $\beta(u,v)$ is denoted by $u \otimes v$ so if $u = \sum_{i=1}^{m} a_i u_i$, $v = \sum_{j=1}^{n} b_j v_j$, then

$$u \otimes v := \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \overbrace{u_i \otimes v_j}^{\text{basis elements.}}.$$

The elements of the form $u \otimes v$ are often called tensor monomials.

Note that, by construction, $(U \otimes V, (u,v) \mapsto u \otimes v)$ depends on the choice of bases in U & V. In the next section, we canonically identify pairs corresponding to different choices.

1.2) Tensor products: universal property.

The main purpose of tensor products is to convert bilinear maps to linear ones. Below is the precise result, the "universal property", that also can be used to show that $U \otimes W$ is independent of choices of bases.

Let U, V, W, W' be vector spaces & $\beta: U \times V \to W$ be a bilinear map. We note that if $\varphi: W \to W'$ is linear, then $\varphi \circ \beta$ is bilinear. To see this note that $\forall u \in U$, the map

$$\varphi \circ \beta(u, ?): V \to W'$$

is linear as the composition of linear maps $V \xrightarrow{\beta(u,?)} W \xrightarrow{\varphi} W'$ (and the same is true when we fix $v \in V$).

Proposition: $\forall$ bilinear map $\beta: U\times V \to W$ $\exists!$ linear map $\tilde\beta: U\otimes V \to W$ s.t. $\tilde\beta(u\otimes v)=\beta(u,v)$ $\forall\, u\in U, v\in V$.

Proof: We must have $\tilde\beta(u_i\otimes v_j)=\beta(u_i,v_j)$. Since the elements $u_i\otimes v_j$ form a basis in $U\otimes V$, this determines $\tilde\beta$ uniquely. Note that both maps $(u,v)\mapsto\beta(u,v)$ & $(u,v)\mapsto\tilde\beta(u\otimes v)$ are bilinear. They coincide on the pairs of basis elements (u_i,v_j) and so coincide everywhere by Remark in Sec 2.1 of Lec 3. $\square$

Now we establish the independence of the choice of bases.

Corollary: Let $u_i'\in U$, $i=1,\dots m$, $v_j'\in V$, $j=1,\dots,n$, be another pair of bases and $(U\otimes' V, (u,v)\mapsto u\otimes' v)$ be the corresponding tensor product. Then there is the unique vector space isomorphism $\iota: U\otimes V \xrightarrow{\sim} U\otimes' V$ satisfying $\iota(u\otimes v)=u\otimes' v$.

Proof: Define $\beta: U\times V\to U\otimes' V$ by $\beta(u,v)=u\otimes' v$ and set $\iota:=\tilde\beta$ so that $\iota(u\otimes v)=u\otimes' v$ $\forall\, u\in U, v\in V$. This determines ι uniquely by Proposition. It remains to show ι is an isomorphism. For this we produce the inverse. Similarly to ι, we get $\iota': U\otimes' V \to U\otimes V$ w. $\iota'(u\otimes' v) = u\otimes v$. We then get $\iota\iota'(u\otimes' v) = u\otimes' v$, $\iota'\iota(u\otimes v) = u\otimes v$. The vectors $u\otimes v$ include a basis so $\iota'\iota = Id_{U\otimes V}$. Similarly, $\iota\iota' = Id_{U\otimes' V}$. $\square$

So we have the well-defined notion of the tensor product. Usually, we abuse the terminology and say that $U\otimes V$ is the tensor product thus omitting the bilinear map $(u,v)\mapsto u\otimes v$ (although it is the most important part of the structure).

1.3) Tensor products, duals, and Hom's.

Recall that for a vector space U over $\mathbb{F}$ we can consider its dual $U^* := Hom(U,\mathbb{F})$, i.e. the space of linear functions $U\to\mathbb{F}$. The goal of this section is to identify $U^*\otimes V$ w. $Hom(U,V)$ for finite dimensional spaces U & V.

By Proposition in Sec 2, to construct a linear map $U^*\otimes V \longrightarrow Hom(U,V)$ amounts to constructing a bilinear map

$U^* \times V \longrightarrow \mathrm{Hom}(U,V)$. For $\alpha \in U^*, v \in V$, define $\varphi_{\alpha,v}: U \to V$ by $\varphi_{\alpha,v}(u) = \alpha(u)v$. We claim that $\varphi_{\alpha,v}$ is linear and the map $(\alpha, v) \mapsto \varphi_{\alpha,v}$ is bilinear. This can be done by a direct check, left as exercise. An alternative way to construct $\varphi_{\alpha,v}$ is by identifying V w. $\mathrm{Hom}(\mathbb{F}, V)$ – to v we assign the map $a \mapsto av$: $\mathbb{F} \to V$ and observe that $\varphi_{\alpha,v}$ is the composition $U \xrightarrow{\alpha} \mathbb{F} \xrightarrow{v} V$. And the map of taking the composition of linear maps is bilinear (compare to Example in Sec 2.1 of Lec 3)

Lemma: The linear map $U^* \otimes V \longrightarrow \mathrm{Hom}(U,V)$ with $\alpha \otimes v \mapsto \varphi_{\alpha,v}$ (that exists and is unique by Proposition in Sec. 1.2) is a vector space isomorphism.

Proof: Choose bases $u_1, \ldots u_m \in U$, $v_1, \ldots v_n \in V$. The choice of a basis in U gives the so called <u>dual basis</u> $\alpha_1, \ldots, \alpha_m \in U^*$ defined by $\alpha_i(u_k) = \delta_{ik}$. These choices give a basis $\alpha_i \otimes v_j \in U^* \otimes V$ and identify $\mathrm{Hom}(U,V)$ w. the space of $n \times m$ matrices. Under this isomorphism φ_{α_i, v_j} is the matrix unit E_{ji}. They form a basis in $\mathrm{Hom}(U,V)$

and our claim about an isomorphism follows. $\square$

Remark: Note that $\varphi_{\alpha,v}$'s are exactly rk 1 linear maps.

1.4) "Algebra" of tensor products

A fun (and important) fact is that vector spaces w.r.t. operations $\oplus, \otimes$ behave like elements of a commutative associative ring:

Proposition: Let U, V, W be (finite dimensional) vector spaces. We have the following "natural" vector space isomorphisms:

- $(U\otimes V)\otimes W \xrightarrow{\sim} U\otimes(V\otimes W)$, unique s.t. $(u\otimes v)\otimes w \mapsto u\otimes(v\otimes w)$.
- $U\otimes V \xrightarrow{\sim} V\otimes U$, unique s.t. $u\otimes v \mapsto v\otimes u$.
- $(U\oplus V)\otimes W \xrightarrow{\sim} U\otimes W \oplus V\otimes W$, unique s.t. $(u,v)\otimes w \mapsto (u\otimes w, v\otimes w)$
- $\mathbb{F}\otimes V \xrightarrow{\sim} V$, unique s.t. $a\otimes v \mapsto av$.

Proof: an exercise. Hint: the maps are uniquely determined by where they send basis elements. Then we use the bilinearity to compute them on the general tensor monomials. $\square$

In what follows we will always identify the tensor products in the proposition by means of these isomorphisms.

2) Tensor products and duals of group representations.

Our goal in this section is to upgrade some constructions of Section 1 to representations of groups.

2.1) Tensor products of representations.

Proposition: Let U, V be representations of a group G. Then there's the unique structure of a representation of G in $U\otimes V$ s.t.

$$g_{U\otimes V}(u\otimes v) = (g_U u)\otimes(g_V v) \quad \forall u\in U, v\in V. \qquad (*)$$

Proof: We start by showing that $\exists!$ linear $g_{U\otimes V}: U\otimes V\to U\otimes V$ satisfying $(*)$. The map $(u,v)\mapsto (g_U u)\otimes(g_V v): U\times V\to U\otimes V$ is bilinear (exercise, hint: g_U & g_V are linear, compare to Sec 1.2)

This shows that $\exists!\ g_{U\otimes V}$. To show that $g\mapsto g_{U\otimes V}$ is a representation of G, it's enough to show that $e_{U\otimes V} = Id_{U\otimes V}$ &

$$(gh)_{U\otimes V} = g_{U\otimes V}\circ h_{U\otimes V} \quad \forall g,h\in G. \qquad (*)$$

These imply that $g_{U\otimes V}$ is invertible (compare to Sec 2.4, Lec 3).
The equality $e_{U\otimes V} = Id_{U\otimes V}$ is easy. To check (*) note that

$$g_{U\otimes V} \circ h_{U\otimes V}(u\otimes v) = g_{U\otimes V}((h_U u)\otimes(h_V v)) = (g_U h_U u)\otimes(g_V h_V v) =$$
$$= (gh)_{U\otimes V}(u\otimes v) \Rightarrow (gh)_{U\otimes V} = g_{U\otimes V}\circ h_{U\otimes V} \qquad \square$$

Not surprisingly, we call $U\otimes V$ the tensor product representation.

2.2) Duals & Hom.

We define the following representations of G.

- The trivial representation in $\mathbb{F}$ (w. $g \mapsto 1 \ \forall g\in G$)
- For representations U, V of G, the representation in $Hom(U,V)$ by $g.\varphi := g_V \circ \varphi \circ g_U^{-1}$ $(g\in G, \varphi \in Hom(U,V))$, Hom representation.
  To check this is indeed a representations is left as an exercise.
- For a representation U of G, the representation in $U^* = Hom(U, \mathbb{F})$ (where $\mathbb{F}$ is trivial), explicitly $g.\alpha = \alpha \circ g_U^{-1}$. This is the dual representation.

The following lemma summarizes important properties of these representations.

Lemma: a) $\mathrm{Hom}_G(U,V) = \mathrm{Hom}(U,V)^G$

b) The isomorphism $U^*\otimes V \xrightarrow{\sim} \mathrm{Hom}(U,V)$ from Sec 1.3 is an isomorphism of representations.

Proof: a): $\varphi\in \mathrm{Hom}_G(U,V) \Leftrightarrow g_V\circ\varphi = \varphi\circ g_U\ \forall g \Leftrightarrow g_V\circ\varphi\circ g_U^{-1}=\varphi$ $\Leftrightarrow \varphi\in \mathrm{Hom}(U,V)^G$.

b) Note that $g.(\alpha\otimes v) = (\alpha\circ g_U^{-1})\otimes(g_V v)$. So we need to check that

$$g_V\circ\varphi_{\alpha,v}\circ g_U^{-1} = \varphi_{\alpha\circ g_U^{-1},\, g_V v}$$

This check is left as an exercise $\square$

2.3) What about representations of associative algebras?

In Sec 2.4 of Lec 3 we have learned that a representation of G is the same thing as a representation of a suitable associative algebra, the group algebra $\mathbb{F}G$. Note, however that the constructions of this section do not make sense for representations of an arbitrary associative algebra (which may fail to have any 1-dimensional representations, for example). There's an additional <u>structure</u> on an associative algebra that enables these constructions ("Hopf algebra") and $\mathbb{F}G$ comes w. such thanks to its construction.