Lecture 5: Irreducible & completely reducible

1) Irreducible representations.

We proceed to studying certain classes of representations: "irreducible" & "completely reducible" ones.

Our general setting is that $\mathbb{F}$ is a field & $A$ is an associative (unital) $\mathbb{F}$-algebra. As in Sec 2 of Lec 1, we say that a representation $V$ of $A$ is irreducible if it's $\neq \{0\}$ and contains no proper subrepresentations i.e subrepresentations $U$ different from $\{0, V\}$.

Example: Let $A = End(V)$ so that $V$ is a representation of $A$ - Example 1 in Sec 2.3 of Lec 3. It's irreducible. Indeed, let $\{0\} \neq U \subset V$ be a subrepresentation. Pick $0 \neq u \in U$. Then $\forall$
$v \in U \Rightarrow \exists \varphi \in \text{End}(V) | \varphi u = v \Rightarrow v \in U \Rightarrow U = V.$

11) Example: 1-dimensional representations of groups.

Any 1-dimensional representation $V$ is irreducible - there are no proper subspaces. Here we will study 1-dimensional representations of $G$ so that $A = GF$.

A choice of basis (= a nonzero element in $V$) identifies $GL(V)$ with $GL(F) = GF \setminus \{0\}$ (w.r.t. multiplication). This identification is independent of the choice - the group $GF \setminus \{0\}$ is abelian, so the conjugation is the identity automorphism of $GF \setminus \{0\}$. Hence a 1-dimensional representation of $G$ is a group homomorphism $G \to GF \setminus \{0\}$.

Recall that for $g, h \in G$, we can define their commutator $(g, h) := g h g^{-1} h^{-1}$. The subgroup generated by these elements is normal, it's called the derived subgroup of $G$ and is denoted by $(G, G)$.

Since $GF \setminus \{0\}$ is abelian, $\varphi((g, h)) = 1$ and hence $\varphi$ factors through the quotient $G/(G, G)$, an abelian group. We arrive at a bijection between the 1-dimensional representations of $G$ & of $G/(G, G)$ (any representation of $G/(G, G)$ can be viewed as that
of $G$ via the pullback w.r.t. $G \rightarrow C/(G,G)$, Sec 2.4 of Sec 2.
Below we assume $G$ is abelian and finite.

Recall that $G \cong \prod_{i=1}^{k} (\mathbb{Z}/m_i \mathbb{Z})$ for some $k \geq 0$, $m_1, \ldots, m_k > 1$. We note that for any abelian groups $G_1, \ldots, G_k$, $H$ we have a bijection

\[ \text{Hom}_{\mathbb{Z}/r} \left( \prod_{i=1}^{k} G_i, H \right) \cong \prod_{i=1}^{k} \text{Hom}_{\mathbb{Z}/r} (G_i, H), \]

where $\text{Hom}_{\mathbb{Z}/r}$ denotes the set of groups homomorphisms, and the map sends $\phi$ to $(\phi|_{G_1}, \phi|_{G_2}, \ldots, \phi|_{G_k})$ (exercise). This reduces the question of describing the 1-dimensional representations of finite (abelian) groups to that for cyclic, which is handled (in the case when $\mathbb{F}$ is algebraically closed) by Prob 1 of HW 1: a homomorphism $\mathbb{Z}/m \mathbb{Z} \rightarrow \mathbb{F}^* \{0\}$ is unique recovered from the image of a generator, which can be any $m$th root of 1 so that we get $m$ pairwise non-isomorphic representations (you still need to work out the details).

1.2) Some irreducible representations of symmetric groups

Let $G = S_n$ (and $A = \mathbb{F} S_n$). We are going to construct some irreducible representations of $G$. Let’s start w. 1-dimensional.
Example: Define the sign representation of $S_n$ as the homomorphism $S_n \rightarrow \mathbb{F}\{0,1\}$ given by $g \mapsto \text{sgn}(g)$.

Exercise: The 1-dimensional representation of $S_n$ is isomorphic to the trivial or sign (hint: either prove that $(S_n, S_n)$ is the subgroup of even permutations or observe that all transpositions $(ij)$ go to the same element of $\mathbb{F}\{0,1\}$ and use $(ij)^2 = e$ to show this element $= \pm 1$. Moreover, if char $\mathbb{F} \neq 2$, then trivial and sign representations are not isomorphic.

Now we proceed to irreducible representations of higher dimensions. Recall the permutation representation: $V = \mathbb{F}^n$ w. $S_n$ acting by permuting the coordinates: $g \cdot (a_1, \ldots, a_n) = (a_{g^{-1}(1)}, \ldots, a_{g^{-1}(n)})$ (Example 1 in Sec 1 of Lec 1). As any Fun representation, it has two subrepresentations, to be denoted here by $F_{\text{const}} = \{ (a, \ldots, a) \}$ (trivial as a representation) and $F_0 = \{ (a_1, \ldots, a_n) \mid \sum_{i=1}^n a_i = 0 \}$. The following lemma establishes some properties of these representations.
Lemma: 1) We have $\mathbb{F}_c^n \cong \mathbb{F}_c^n$ iff char $\mathbb{F}$ divides $n$.

2) Otherwise, $\mathbb{F}_c^n \oplus \mathbb{F}_c^n = \mathbb{F}_c^n$.

3) and $\mathbb{F}_c^n$ is irreducible.

4) $\text{sgn}_n \otimes \mathbb{F}_c^n$ is irreducible iff $\mathbb{F}_c^n$ is.

5) $\text{sgn}_n \otimes \mathbb{F}_c^n$ is not isomorphic to $\mathbb{F}_c^n$ for $n \neq 3$.

Proof: 1) & 2) are left as exercises.

3) Set $e_i = (0, 0, 1, -1, 0, \ldots, 0)$, $i = 1, \ldots, n-1$. These vectors form a basis in $\mathbb{F}_c^n$. If a subrepresentation $U$ contains one of them, it contains all: $6e_i = e_j$ for $6 \in S_n$ with $6(i) = 6(j), 6(i+1) = 6(j+1)$.

$\text{char } \mathbb{F}$ doesn't divide $n \Rightarrow \forall \; \mathbf{v} = (x_1, \ldots, x_n) \in \mathbb{F}_c^n \setminus \{0\} \exists \; i \mid x_i \neq x_{i+1}$. Then $\mathbf{v} - (i, i+1). \mathbf{v} = (x_i - x_{i+1}) e_i$, so $\mathbf{v} \in U \Rightarrow e_i \in U \Rightarrow U = \mathbb{F}_c^n$.

4) We prove a more general claim: if $W$ & $V$ are representations of $G$ & $\dim W = 1$, then $V$ is irreducible iff $W \otimes V$ is. Let $\rho_W : G \to GL(W)$, $\rho_V : G \to GL(V)$ be the corresponding homomorphisms. We can identify $W$ w. $\mathbb{F}$ and hence $W \otimes V = \mathbb{F} \otimes V$. 

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V as vector spaces (Sec 1.4 of Lec 9). Under this identification, $p_{w \circ v}(g) = p_w(g)p_v(g)$.

So a subspace in $W \otimes V$ is a subrepresentation ($\Leftrightarrow$ stable under all $p_{w \circ v}(g)$) iff the same subspace in $V$ is stable under all $p_v(g)$ - any subspace is stable under scalar operators.

5) Exercise. Hint: $(1,2)$ has eigenvalues $-1$ w. multiplicity 1 and 1 w. multiplicity $n-2$ on $\mathbb{F}_n^*$ and vice versa on $sgn^* \otimes \mathbb{F}_n^*$. □

2) Completely reducible representations & Maschke’s Thm.

Definition: Let $A$ be an associative algebra over $\mathbb{F}$. An $A$-module $V$ is called completely reducible if it submodule $U \subseteq V \exists$ a complement: submodule $U' \subseteq V$ w. $U \oplus U' = V$ (the direct sum of subspaces).

Note that then $V \cong U \oplus U'$ as $A$-modules. Note also that every irreducible representation $V$ is completely reducible - just two options for $U \subseteq V$: $\{0\}$ and $V$. 6
The same definition applies to representations of a group $G$ because a representation of $G$ is the same thing as a representation of the group algebra $FG$.

The following result is very important and gives motivation to consider completely reducible representations.

Thm (Maschke): Let $|G| < \infty$. Further, assume that either $\text{char } F = 0$ or $\text{char } F > 0$ but $\text{char } F \nmid |G|$. Then every (finite dimensional) representation of $G$ over $IF$ is completely reducible.

We'll prove this theorem in the next lecture. We'll also see that this statement implies one from Sec 2 of Lec 1.

Remark: Let's see how the conclusion of the theorem fails if one of the conditions doesn't hold:

1) Suppose $p = \text{char } IF$ divides $n$, let $G = S_n$ & $V = IF^n$, $U = IF^n$. A complement $U'$ must have dim $= 1$. So $U' = IF \oplus$ with
\[ u = (a_1, \ldots, a_n). \] Assume \( n > 2 \) (the case \( n = 2 \) is left as an exercise).

\( g \in S_n \) w. \( g^2 = 1 \) (i.e. \( g = (i, j) \)) we have \( g v = \pm v \). One can check that this implies \( a_1 = \ldots = a_n \). A contradiction: \( v \in F^n / b/c \ p \mid n \).

2) Consider the group \( \mathbb{G}^{2*} = \{ (a, b) \mid a, b \in F \} \subset GL(F) \), infinite if \( F \) is, and its representation in \( F^2 \) given by the inclusion into \( GL(F) \). From \((a, b)(0, 1) = (a + b, b)\), one sees that \( U = \{(a) \mid a \in F\} \) is the only proper subrepresentation. So \( F^2 \) is not completely reducible.

2.1) General properties of completely reducible representations.

One can ask whether the complete reducibility is preserved under the natural operations w. modules. The answer is Yes.

Lemma: Let \( V_1, V_2 \) be completely reducible \( A \)-modules. Then

(i) Every submodule \( U_1 \subseteq V_1 \) is completely reducible.

(ii) \( V_1 \oplus V_2 \) is completely reducible.

We'll prove this next time.
Corollary: Let $V$ be a finite dimensional $A$-module. TFAE:

(a) $V$ is completely reducible.

(b) $V$ is isomorphic to the direct sum of irreducible modules.

Proof: (a) $\Rightarrow$ (b) is exercise - use induction on $\dim V$ & i) of Lemma.

(b) $\Rightarrow$ (a) follows from ii) of Lemma by induction on the number of summands. □

The corollary allows to reduce the study of completely reducible modules to the study of irreducible ones.