Lecture 5: Irreducible & completely reducible representations, pt 1.

1) Irreducible representations.
2) Completely reducible representations.

Ref: Sec 11.1 in [V].

1) Irreducible representations.

We proceed to studying certain classes of representations: "irreducible" & "completely reducible" ones.

Our general setting is that $\mathbb{F}$ is a field & A is an associative (unital) $\mathbb{F}$-algebra. As in Sec 2 of Lec 1, we say that a representation V of A is irreducible if it's $\neq \{0\}$ and contains no proper subrepresentations i.e subrepresentations U different from $\{0\}, V$.

Example: Let $A = End(V)$ so that V is a representation of A – Example 1 in Sec 2.3 of Lec 3. It's irreducible. Indeed, let $\{0\} \neq U \subset V$ be a subresentation. Pick $0 \neq u \in U$. Then $\forall$

$v \in U$ $\exists$ $\varphi \in End(V) | \varphi u = v \Rightarrow v \in U \Rightarrow U = V$.

1.1) Example: 1-dimensional representations of groups.

Any 1-dimensional representation V is irreducible - there are no proper subspaces. Here we will study 1-dimensional representations of G so that $A = \mathbb{F}G$.

A choice of basis (= a nonzero element in V) identifies $GL(V)$ w. $GL_1(\mathbb{F}) = \mathbb{F} \setminus \{0\}$ (w.r.t. multiplication). This identification is independent of the choice - the group $\mathbb{F} \setminus \{0\}$ is abelian, so the conjugation is the identity automorphism of $\mathbb{F} \setminus \{0\}$. Hence a 1-dimensional representation of G is a group homomorphism $G \xrightarrow{\rho} \mathbb{F} \setminus \{0\}$.

Recall that for $g, h \in G$ we can define their commutator $(g,h) := ghg^{-1}h^{-1}$. The subgroup generated by these elements is normal, it's called the derived subgroup of G and is denoted by (G,G).

Since $\mathbb{F} \setminus \{0\}$ is abelian, $\rho((g,h)) = 1$ and hence ρ factors through the quotient $G/(G,G)$, an abelian group. We arrive at a bijection between the 1-dimensional representations of G & of $G/(G,G)$ (any representation of $G/(G,G)$ can be viewed as that of G via the pullback w.r.t. $G \twoheadrightarrow G/(G,G)$, Sec 2.4 of Lec 2)

Below we assume G is abelian and finite.

Recall that $G \simeq \prod_{i=1}^{k} (\mathbb{Z}/m_i\mathbb{Z})$ for some $k \geqslant 0$, $m_1, \ldots m_k > 1$. We note that for any <u>abelian</u> groups $G_1, \ldots G_k, H$ we have a bijection

$$\mathrm{Hom}_{Gr}\left(\prod_{i=1}^{k} G_i, H\right) \xrightarrow{\sim} \prod_{i=1}^{k} \mathrm{Hom}_{Gr}(G_i, H),$$

where Hom_{Gr} denotes the set of groups homomorphisms, and the map sends φ to $(\varphi|_{G_1}, \varphi|_{G_2}, \ldots, \varphi|_{G_k})$ (exercise). This reduces the question of describing the 1-dimensional representations of finite (abelian) groups to that for cyclic, which is handled (in the case when $\mathbb{F}$ is algebraically closed) by Prob 1 of HW1: a homomorphism $\mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{F} \setminus \{0\}$ is unique recovered from the image of a generator, which can be any mth root of 1 so that we get m pairwise non-isomorphic representations (you still need to work out the details).

1.2) Some irreducible representations of symmetric groups.

Let $G = S_n$ (and $A = \mathbb{F}S_n$). We are going to construct some irreducible representations of G. Let's start w. 1-dimensionals.

Example: Define the **sign representation** of S_n as the homomorphism $S_n \to \mathbb{F}\setminus\{0\}$ given by $g \mapsto \operatorname{sgn}(g)$.

Exercise: $\forall$ 1-dimensional representation of S_n is isomorphic to the trivial or sign (hint: either prove that (S_n, S_n) is the subgroup of even permutations or observe that all transpositions (i,j) go to the same element of $\mathbb{F}\setminus\{0\}$ and use $(i,j)^2 = e$ to show this element $= \pm 1$. Moreover, if $\operatorname{char}\mathbb{F} \neq 2$, then trivial and sign representations are not isomorphic.

Now we proceed to irreducible representations of higher dimensions. Recall the permutation representation: $V = \mathbb{F}^n$ w. S_n acting by permuting the coordinates: $g.(a_1, \ldots a_n) = (a_{g^{-1}(1)}, \ldots, a_{g^{-1}(n)})$ (Example 1 in Sec 1 of Lec 1). As any Fun representation, it has two subrepresentations, to be denoted here by $\mathbb{F}^n_{const} = \{(a, \ldots, a)\}$ (trivial as a representation) and $\mathbb{F}^n_0 = \{(a_1, \ldots, a_n) \mid \sum_{i=1}^n a_i = 0\}$. The following lemma establishes some properties of these representations.

Lemma: 1) We have $\mathbb{F}^n_{const} \subset \mathbb{F}^n_0$ iff char $\mathbb{F}$ divides n.

2) Otherwise, $\mathbb{F}^n_{const} \oplus \mathbb{F}^n_0 = \mathbb{F}^n$,

3) and $\mathbb{F}^n_0$ is irreducible.

4) $sgn_n \otimes \mathbb{F}^n_0$ is irreducible iff $\mathbb{F}^n_0$ is.

5) $sgn_n \otimes \mathbb{F}^n_0$ is not isomorphic to $\mathbb{F}^n_0$ for $n>3$.

Proof: 1) & 2) are left as exercises.

3): Set $e_i = (0,..,0,\overset{i}{1},-1,0,...0)$, $i=1,...,n-1$. These vectors form a basis in $\mathbb{F}^n_0$. If a subrepresentation U contains one of them, it contains all: $\sigma e_i = e_j$ for $\sigma \in S_n$ w. $\sigma(i)=\sigma(j)$, $\sigma(i+1)=\sigma(j+1)$.

char $\mathbb{F}$ doesn't divide $n \Rightarrow \forall\, v=(x_1,...x_n) \in \mathbb{F}^n_0 \setminus \{0\}\ \exists\, i \mid x_i \neq x_{i+1}$. Then $v-(i,i+1).v = (x_i - x_{i+1})e_i$, so $v \in U \Rightarrow e_i \in U \Rightarrow U = \mathbb{F}^n_0$.

4) We prove a more general claim: if W & V are representations of G & $\dim W = 1$, then V is irreducible iff $W \otimes V$ is. Let $\rho_W: G \to \mathbb{F}\setminus\{0\}$, $\rho_V: G \to GL(V)$ be the corresponding homomorphisms. We can identify W w. $\mathbb{F}$ and hence $W\otimes V = \mathbb{F}\otimes V$ w.

V as vector spaces (Sec 1.4 of Lec 4). Under this identification

$$\rho_{W\otimes V}(g) = \underset{\text{scalar}}{\rho_W(g)}\rho_V(g).$$

So a subspace in $W\otimes V$ is a subrepresentation ($\Leftrightarrow$ stable under all $\rho_{W\otimes V}(g)$) iff the same subspace in V is stable under all $\rho_V(g)$ - any subspace is stable under scalar operators.

5) Exercise. Hint: $(1,2)$ has eigenvalues -1 w. multiplicity 1 and 1 w. multiplicity $n-2$ on $\mathbb{F}_0^n$ and vice verse on $sgn_n\otimes\mathbb{F}_0^n$. $\square$

2) Completely reducible representations & Maschke's Thm.

Definition: Let A be an associative algebra over $\mathbb{F}$. An A-module V is called **completely reducible** if $\forall$ submodule $U\subset V$ $\exists$ a **complement**: <u>submodule</u> $U'\subset V$ w. $U\oplus U'=V$ (the direct sum of subspaces).

Note that then $V\simeq U\oplus U'$ as A-modules. Note also that every irreducible representation V is completely reducible - just two options for $U\subset V$: $\{0\}$ and V.

The same definition applies to representations of a group G because a representation of G is the same thing as a representation of the group algebra $\mathbb{F}G$.

The following result is very important and gives a motivation to consider completely reducible representations.

Thm (Maschke): Let $|G|<\infty$. Further, assume that either $\operatorname{char}\mathbb{F}=0$ or $\operatorname{char}\mathbb{F}>0$ but $\operatorname{char}\mathbb{F}\nmid|G|$. Then every (finite dimensional) representation of G over $\mathbb{F}$ is completely reducible.

We'll prove this theorem in the next lecture. We'll also see that this statement implies one from Sec 2 of Lec 1.

Remark: Let's see how the conclusion of the theorem fails if one of the conditions doesn't hold:

1) Suppose $p=\operatorname{char}\mathbb{F}$ divides n, let $G=S_n$ & $V=\mathbb{F}^n$, $U=\mathbb{F}^n_0$. A complement U' must have $\dim=1$. So $U'=\mathbb{F}v$ with

$v=(a_1,\dots a_n)$. Assume $n>2$ (the case $n=2$ is left as an exercise). $\forall\, g\in S_n$ w. $g^2=1$ (i.e $g=(i,j)$) we have $gv=\pm v$. One can check that this implies $a_1=\dots=a_n$. A contradiction: $v\in \mathbb{F}_0^n$ b/c $p|n$.

2) Consider the group $G=\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix}\middle|\, x\in\mathbb{F}\right\}\subset GL_2(\mathbb{F})$, infinite if $\mathbb{F}$ is, and its representation in $\mathbb{F}^2$ given by the inclusion into $GL_2(\mathbb{F})$. From $\begin{pmatrix}1&x\\0&1\end{pmatrix}.\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a+xb\\b\end{pmatrix}$, one sees that $U=\left\{\begin{pmatrix}a\\0\end{pmatrix}\middle|\, a\in\mathbb{F}\right\}$ is the only proper subrepresentation. So $\mathbb{F}^2$ is not completely reducible.

2.1) General properties of completely reducible representations.

One can ask whether the complete reducibility is preserved under the natural operations w. modules. The answer is Yes.

Lemma: Let V_1, V_2 be completely reducible A-modules. Then
(i) Every submodule $U_1\subset V_1$ is completely reducible.
(ii) $V_1\oplus V_2$ is completely reducible.

We'll prove this next time.

Corollary: Let V be a finite dimensional A-module. TFAE:

(a) V is completely reducible.

(b) V is isomorphic to the direct sum of irreducible modules.

Proof: (a) $\Rightarrow$ (b) is exercise - use induction on $\dim V$ & i) of Lemma.

(b) $\Rightarrow$ (a) follows from ii) of Lemma by induction on the number of summands. $\square$

The corollary allows to reduce the study of completely reducible modules to the study of irreducible ones.