Bonus Lecture 6.5:

The power of averaging: finite generation of the invariants.

Prerequisite: MATH 380.

1) Main result

Let G be a group and V be a finite dimensional representation of G over $\mathbb{C}$. We can talk about the algebra of polynomial functions $\mathbb{C}[V]$ and its subalgebra of invariants $\mathbb{C}[V]^G$. We are interested in sufficient conditions for $\mathbb{C}[V]^G$ to be finitely generated.

Suppose that we have a $\mathbb{C}[V]^G$-linear map $\alpha: \mathbb{C}[V] \to \mathbb{C}[V]^G$ w. $\alpha(1)=1$.

This is the case when G is finite, we have $\alpha(f):=\varepsilon.f$, where ε is the averaging idempotent, $\varepsilon = \frac{1}{|G|}\sum_{g\in G} g$. This is the case when G is compact but this requires some important discussion.

Note that $\mathbb{C}[V]$ is a graded algebra: the homogeneous degree k component, to be denoted by $\mathbb{C}[V]_k$, is the space of degree k homogeneous polynomials. Note that $\dim \mathbb{C}[V]_k < \infty$ and that $\mathbb{C}[V]_k \subset \mathbb{C}[V]$ is a G-subrepresentation. It follows that $\mathbb{C}[V]^G = \bigoplus_{k \geqslant 0} \mathbb{C}[V]^G_k$ is a graded subalgebra.

Now, back to the case when G is compact. All representations in $\mathbb{C}[V]_k$ are finite dimensional & continuous. So we have the averaging operator $\varepsilon: \mathbb{C}[V]_k \longrightarrow \mathbb{C}[V]^G_k$, see Sec 3 in Lec 6

We define α on $\mathbb{C}[V]_k$ to be this ε. One can show α is $\mathbb{C}[V]^G$-linear (exercise).

The same construction works for reductive groups (such as $GL_n(\mathbb{C})$) and their rational representations.

Now we proceed to the main result.

Thm (essentially, Hilbert) If α like above exists, then $\mathbb{C}[V]^G$ is a finitely generated algebra.

2) Proof.

The proof is in three steps. Set $\mathbb{C}[V]^G_+ := \bigoplus_{k>0} \mathbb{C}[V]^G_k$, this is an ideal in $\mathbb{C}[V]^G$. Let $(\mathbb{C}[V]^G_+)$ denote the ideal in $\mathbb{C}[V]$ generated by $\mathbb{C}[V]^G_+$.

Step 1: Show that $\mathbb{C}[V]^G$ is finitely generated (as an algebra) if $\mathbb{C}[V]^G_+$ is finitely generated as an ideal.

Step 2: Show that $\mathbb{C}[V]^G_+$ is finitely generated (as an ideal in $\mathbb{C}[V]^G$) iff $(\mathbb{C}[V]^G_+)$ is finitely generated (as an ideal in $\mathbb{C}[V]$).

This is where we use the operator α.

Step 3: Use the Hilbert basis theorem to conclude that any ideal in $\mathbb{C}[V]$ (= the algebra of polynomials) is finitely generated - including $(\mathbb{C}[V]^G_+)$. This will complete the proof.

Step 1: We can pick a finite collection $f_1, \dots, f_m$ of <u>homogeneous</u> generators of $\mathbb{C}[V]^G_+$. Then they generate $\mathbb{C}[V]^G$ as an algebra (exercise).

Step 2: Suppose $(\mathbb{C}[V]^G_+)$ is finitely generated. Then we can choose a finite collection of generators from $\mathbb{C}[V]^G_+$, denote it by $F_1, \dots F_k$. We claim that $F_1, \dots F_k$ generate $\mathbb{C}[V]^G_+$. Indeed, pick $F \in \mathbb{C}[V]^G_+$. Since $F \in (\mathbb{C}[V]^G_+)$, $\exists\, h_1, \dots, h_k \in \mathbb{C}[V]$ w.

$$(*) \qquad F = \sum_{i=1}^{k} h_i F_i .$$

Now apply $\alpha : \mathbb{C}[V] \longrightarrow \mathbb{C}[V]^G$ to both sides. On the l.h.s. we have $\alpha(F) = F\alpha(1) = F$. On the r.h.s.: $\alpha\left(\sum_{i=1}^{k} h_i F_i\right) = \sum_{i=1}^{k} \alpha(h_i) F_i$. Since $\alpha(h_i) \in \mathbb{C}[V]^G$, we are done.