Lecture 6: Irreducible & completely reducible representations, pt 2.

1) Proof of Maschke's theorem.

2) Decomposition into irreducibles & Schur lemma.

Ref: Secs 11.1 & 11.2 in [V]; Secs 2.3, 4.1, 4.6 in [E].

1.0) Recap

Our goal in this section is to prove:

Thm (Maschke): Let $|G|<\infty$. Further, assume that either char $\mathbb{F}=0$ or char $\mathbb{F}>0$ but char $\mathbb{F} \nmid |G|$. Then every (finite dimensional) representation V of G over $\mathbb{F}$ is completely reducible.

We also need to prove a lemma from the last time.

Lemma: Let V_1, V_2 be completely reducible A-modules. Then

(i) Every submodule $U_1 \subset V_1$ is completely reducible.

(ii) $V_1 \oplus V_2$ is completely reducible.

Proof: i) Let $U \subset U_1$ be a submodule. Since V_1 is completely reducible $\exists$ submodule $U' \subset V_1$ w. $V_1 = U \oplus U'$. Clearly $U \cap (U' \cap U_1) = \{0\}$; to show $U_1 = U + (U' \cap U_1)$, note that $\forall u_1 \in U_1$ $\exists u \in U$, $u' \in U'$ s.t. $u_1 = u + u'$ (from $V_1 = U + U'$). Then $u' = u_1 - u \in U_1 \cap U'$, i.e. $U_1 = U + (U' \cap U_1) \Rightarrow U_1 = U \oplus (U' \cap U_1)$.

ii) Let $U \subset V_1 \oplus V_2$ be a submodule. We'll find submodules $U_i \subset V_i$ s.t. $U \oplus (U_1 \oplus U_2) = V_1 \oplus V_2$.

Consider the projection $V_1 \oplus V_2 \to V_2$, $(v_1, v_2) \mapsto v_2$ and let π be its restriction to U. In particular, $\pi : U \to V_2$ is an A-module homomorphism. So, $\ker \pi (= U \cap V_1) \subset U$ & $\operatorname{im} \pi \subset V_2$ are submodules. Since V_1, V_2 are completely reducible, we can find submodules $U_1 \subset V_1$ s.t. $U_1 \oplus (U \cap V_1) = V_1$ & $U_2 \subset V_2$ s.t. $U_2 \oplus \operatorname{im} \pi = V_2$. We claim that $U \oplus (U_1 \oplus U_2) = V_1 \oplus V_2$.

Let $u_i \in U_i$, $i = 1, 2$, be s.t. $u_1 + u_2 \in U$. Note that $u_2 = \pi(u_1 + u_2) \in \operatorname{im} \pi \Rightarrow u_2 \in \operatorname{im} \pi \cap U_2 = \{0\}$. Then $u_1 \in (V_1 \cap U) \cap U_1 = \{0\}$. We see that $U \cap (U_1 \oplus U_2) = \{0\}$.

Now we prove $U + (U_1 \oplus U_2) = V_1 \oplus V_2$. Pick $v_i \in V_i$. We can find $u' \in U$ & $u_2 \in U_2 \mid v_2 = \pi(u') + u_2$. Then $v_1 + v_2 - u' - u_2 \in V_1$ and we can find $u'' \in V_1 \cap U$ & $u_1 \in U_1 \mid v_1 + v_2 - u' - u_2 = u'' + u_1 \Rightarrow$

$v_1+v_2=(u'+u'')+(u_1+u_2)$, the desired decomposition. $\square$

Note that thx to Corollary in Sec 2.1 of Lec 5 (which followed from the lemma we've just proved), every completely reducible representation is isomorphic to the direct sum of irreducible representations. The theorem therefore reduces the study of completely reducible representations to that of irreducibles.

1.1) Averaging

We will give one proof of Maschke's thm based on "invariant projectors" and sketch an alternative proof for $\mathbb{F}=\mathbb{C}$ based on "invariant Hermitian scalar products." Both are based on the same key idea - averaging, to be explained in this section.

Under the assumption of the theorem, the following element of $\mathbb{F}G$ makes sense: $\varepsilon=\frac{1}{|G|}\sum_{g\in G} g$. It's called the averaging (or trivial) idempotent. Here are its basic properties

Lemma: a) $h\varepsilon=\varepsilon \ \forall h\in G$.

b) Let V be a representation of G. For any $v \in V$, the element $\varepsilon . v \in V^G$ (invariants),

c) & if $v \in V^G$, then $\varepsilon . v = v$.

Proof: a) follows by reordering the summands, b) follows from a) & c) is clear. Details are exercise. $\square$

Rem: $\varepsilon^2 = \varepsilon$ – which is why ε is called an idempotent.

1.2) Proof via invariant projectors.

We need an equivalent formulation of the existence of a G-stable complement. For this we need projectors.

Definition: Let V be a vector space over $\mathbb{F}$ & $U \subset V$ be a subspace. By a projector to U we mean an operator $P \in \mathrm{End}(V)$ s.t.

(i) $\operatorname{im} P \subset U$.

(ii) $P(u) = u$, $\forall\, u \in U$.

Note that $\operatorname{im} P = U$ & $P^2 = P$.

Lemma: There is a bijection between:

(a) the projectors to U.

(b) the complements to U, i.e. subspaces $U' \subset V$ w. $U \oplus U' = V$.

Proof:

(b) $\to$ (a): Set $P_{U'}(u+u') := u$, $u \in U$, $u' \in U'$; $P_{U'}$ satisfies (i) & (ii).

(a) $\to$ (b): Set $U'_P = \ker P$. Note that $P(v - P(v)) = P(v) - P^2(v) = 0$. So $U + U'_P = V$. Since $P(v) = v$ $\forall v \in U$ & $P(v) = 0$ $\forall v \in U'_P$, we get $U \cap U'_P = \{0\}$, so $U \oplus U'_P = V$.

The claim that $U' \mapsto P_{U'}$ & $P \mapsto U'_P$ are inverse to each other is left as an exercise. $\square$

Proof of Theorem: Note that if V is a representation of G, $U \subset V$ a subrepresentation, and a projector P to U is a homomorphism of representations, then $U' = \ker P$ is also a subrepresentation (Lemma 1 in Sec 1 of Lec 3) and so by the previous lemma, U' is a complement to U.

By a) of Lemma in Sec 2.2 of Lec 4, $\varphi \in End(V)$ is a homomorphism of representations iff it's invariant for the representation of G in $End(V)$ given by $g.\varphi = g_V \circ \varphi \circ g_V^{-1}$).

Pick a projector P to U, at least one exists. Consider

$$\underline{P} := \varepsilon.P \left(= \frac{1}{|G|} \sum_{g \in G} g.P\right),$$

an invariant. It remains to check that $\underline{P}$ is a projector to U.

Check (i): $\underline{P}(v) = \frac{1}{|G|} \sum_{g \in G} g_V \circ P \circ g_V^{-1}(v)$ [(i) for P & $g_V(U) \subset U$] $\in U$.

Check (ii): $u \in U \Rightarrow \underline{P}(u) = \frac{1}{|G|} \sum_{g \in G} g_V \circ P \circ g_V^{-1}(u) =$ [$g_V^{-1}(u) \in U$ & (ii) for P] $= \frac{1}{|G|} \sum_{g \in G} g_V \circ g_V^{-1}(u) = \frac{|G|}{|G|} u = u.$ $\square$

1.3) Sketch of proof via invariant Hermitian product.

Let $\mathbb{F} = \mathbb{C}$, and V be a finite dimensional representation of G over $\mathbb{C}$. Recall that by a Hermitian form on V we mean an $\mathbb{R}$-bilinear map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying the following additional conditions:

- It's $\mathbb{C}$-linear in the 1st argument,
- and $\langle u,v\rangle = \overline{\langle v,u\rangle}$ $\forall\, u,v\in V$.

We say that $\langle\cdot,\cdot\rangle$ is a Hermitian scalar product if, in addition, $\langle v,v\rangle > 0$ $\forall\, v\in V, \neq 0$. Such $\langle\cdot,\cdot\rangle$ exists.

For a subspace $U\subset V$ and a Hermitian scalar product $\langle\cdot,\cdot\rangle$ we can consider the orthogonal complement $U^\perp$ so that $U\oplus U^\perp = V$.

Let $\mathrm{Herm}(V)$ denote the set of Hermitian forms, it's a vector space over $\mathbb{R}$ (addition and multiplic'n by scalars of functions).

A proof of Theorem is based on the following exercises.

Exercise 1: a) $\mathrm{Herm}(V)$ is a representation of G (over $\mathbb{R}$) via $g.\langle\cdot,\cdot\rangle$ defined by $(u,v)\mapsto \langle g^{-1}u, g^{-1}v\rangle$.

b) If $U\subset V$ is a subrepresentation, and $\langle\cdot,\cdot\rangle\in \mathrm{Herm}(V)$ is G-invariant, then $U^\perp$ is a subrepresentation.

Exercise 2: If $\langle \cdot, \cdot \rangle$ is a Hermitian scalar product, then so is $\mathcal{E}.\langle \cdot,\cdot\rangle = \frac{1}{|G|}\sum_{g\in G} g.\langle\cdot,\cdot\rangle$.

Thx to Exercise 2, there is a <u>G-invariant</u> Hermitian scalar product. Now we use (6) of Exercise 1 and take $U' = U^{\perp}$.

1.4) Remarks.

1) The 2nd proof we sketched is more narrow in scope. The approach, however, has an advantage, once an invariant Hermitian scalar product $\langle\cdot,\cdot\rangle$ is fixed, we have a preferred way of recovering U' from U. And in some settings we have such $\langle\cdot,\cdot\rangle$. Here are two of them that are beyond the scope of this course:

I) Let G be a (possibly, infinite) group.

Let X be a space with a measure and G act on X by preserving the measure. Then $L^2(X)$ comes w. $\langle\cdot,\cdot\rangle$ - it's a "Hilbert space", and G acts on $L^2(X)$ by unitary operators.

In particular, $L^2(X)$ is completely reducible (in this setting one restricts to subrepresentations that closed w.r.t. the topology given by the norm).

II) Representations in Hilbert spaces by unitary operators are also important for Quantum Mechanics: Hilbert spaces appear as "spaces of states" for quantum mechanical systems and group representations by unitary operators appear as symmetries.

2) The constructions and results of this section extend to some infinite groups (if one suitably restricts the class of representations considered). Most notably, this is the case for continuous representations of compact groups (the summation in the construction of the averaging operator needs to be replaced with an integral). We elaborate on this in the bonus section.

3) Averaging is useful for other purposes as well, e.g. to prove that the algebras of invariants are finitely generated. This is the subject of Bonus lecture 6.5.

2) Decomposition into irreducibles & Schur lemma.

Let A be an associative algebra and V is its finite dimensional representation. Assume V is completely reducible. By Cor in Sec 2.1 of Lec 5, V is isomorphic to $\bigoplus_{i=1}^{k} U_i^{\oplus m_i}$, where $U_1, \dots, U_k$ are pairwise non-isomorphic irreducible representations of A. Our question for now is: how to compute the numbers m_i.

Proposition: $m_i = \dim \mathrm{Hom}_A(U_i, V) / \dim \mathrm{End}_A(U_i)$

The r.h.s. is called the multiplicity of U_i in V. Note that it depends only on U_i, V, not on the choice of decomp'n.

The proof of the proposition is based on the following fundamentally important result.

Theorem (Schur Lemma) Let A be an associative algebra over $\mathbb{F}$ and U, V be irreducible A-modules. Then

a) any A-module homomorphism $\varphi: U \to V$ is either 0 or invertible.

b) Suppose $\mathbb{F}$ is algebraically closed and $\dim V < \infty$. Any homomorphism $\varphi: V \to V$ is proportional to Id_V.

Proof: a) Recall, Lemma 1 in Sec 1 of Lec 3, that $\ker\varphi \subset U$, $\operatorname{im}\varphi \subset V$ are submodules. If $\varphi \neq 0$, then $\ker\varphi \subsetneq U$, $\operatorname{im}\varphi \supsetneq \{0\}$. But U, V are irreducible, so $\ker\varphi = \{0\}$, $\operatorname{im}\varphi = V$. Hence φ is bijective, i.e. invertible.

b) Under our assumptions, φ has an eigenvalue, say a. Note that $\varphi - a\,Id_V$ is also a homomorphism, c) of Lem 2 in Sec 1, Lec 3. Since $\varphi - a\,Id_V$ is not invertible, by a), it's zero $\square$

Rem: This claim is a "lemma" b/c the proof is easy. But it's still a very important basic theorem - we'll see several more applications later.

We'll prove Proposition in the next lecture.

3) Bonus: averaging for infinite groups.

For some infinite groups, one can write down analogs of the operator $v \mapsto \varepsilon v$ for <u>some</u> representations. As for the representations of finite groups, this will show that the representations in this class are completely reducible.

The first example occurs in the world of topological groups and their finite dimensional "continuous" representations $V/\mathbb{C}$: choosing a basis in V, we identify $GL(V)$ w. $GL_n(\mathbb{C})$, so that $\rho: G \to GL_n(\mathbb{C})$ is specified by n^2 functions, matrix coefficients. We say that V is continuous if these functions are.

We need our group to be compact as a topological space. A basic example is U_n the subgroup of unitary matrices in $GL_n(\mathbb{C})$. It's compact b/c it's closed (given by a bunch of equations & bounded (all columns have length 1) in $Mat_n(\mathbb{C})$. The group of real orthogonal matrices, $O_n(\mathbb{R})$ is compact too.

A basic fact about a compact group G is that they have a distinguished measure μ, called Haar measure, that is invariant under left (and right) translations and $\mu(G)=1$.

In the easy special case when the group G is finite, we take the measure w. value $\frac{1}{|G|}$ at every point. For compact Lie groups (such as $U(n)$ or $O(n,\mathbb{R})$), it comes from a suitably normalized left invariant top differential form.

We can integrate continuous functions on G w.r.t. the measure μ. This extends to V-valued functions for any finite dimensional vector space. Now if V is a continuous representation, the map $g \mapsto gv : G \longrightarrow V$ is continuous for any $v \in V$, so can be integrated. We define $\varepsilon_V : V \longrightarrow V$ by

$$\varepsilon_V v := \int gv \, d\mu.$$

Since μ is left invariant, we have $\varepsilon_V v \in V^G$ and since $\int d\mu = 1$, we have $\varepsilon_V v = v \;\forall\; v \in V^G$.

The existence of averaging operators for compact groups yields the existence of averaging operators for "rational representations" of "complex reductive groups." An example is $GL_n(\mathbb{C})$ and a representation is rational if its matrix coefficients are polynomials in the n^2 entries & $\det^{-1}$. All representations obtained from the tautological representation $\mathbb{C}^n$ (corresponding to the identity homomorphism $GL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$) by means of taking direct sums & summands, tensor products and duals are rational.

By definition, the averaging operator for a rational representation is the same as for U_n. The only thing we need to check is that any U_n-invariant vector in a rational representation of $GL_n(\mathbb{C})$ is $GL_n(\mathbb{C})$-invariant. For this one observes that the condition $gv=v$ is equivalent to vanishing of some matrix coefficients on g. Then one uses that U_n is "Zariski dense" in $GL_n(\mathbb{C})$: the only function that is polynomial in the matrix entries and $\det^{-1}$ that vanishes on U_n is 0. This is easy when $n=1$ (any Laurent polynomial in z vanishing for $|z|=1$ is 0) and requires some theory for $n>1$. Passing from $GL_n(\mathbb{C})$ to U_n in this context is known as "unitary trick."