Lecture 7: Irreducible & completely reducible representations, pt 3.

1) Decomposition into irreducibles, cont'd.
2) Decomposition of regular representation & applications.
3) Skew-fields.

Ref: Secs 11.1, 11.2, 11.4, 11.6 in [V].

0) Recap and goals.

In Sec 1.2 of Lec 6 we have proved the Maschke thm that states, in particular, that every finite dimensional representation V of a finite group G over a characteristic 0 field $\mathbb{F}$ is completely reducible. This motivates us to consider completely reducible representations – even in the situations when not all representations are completely reducible.

Let A be an associative algebra, and V its completely reducible representation. In Sec 2 of Lec 6 we've stated that

(1) $V = \bigoplus_{i=1}^{k} U_i^{\oplus m_i}$, $m_i = \dim \mathrm{Hom}_A(U_i, V)/\dim \mathrm{End}_A(U_i)$,

where U_i's are some pairwise nonisomphic irreducible rep's of A.

Note that in (1) we allow some m_i to be 0.

We'll deduce (1) today using the Schur lemma (proved in Sec 2 of Lec 6)

Thm: Let U, V be irreducible representations of A.

(a) Any A-module homomorphism $\varphi: U \to V$ is either 0 or is invertible.

(b) Suppose $\mathbb{F}$ is algebraically closed, and $\dim U < \infty$. Then any A-module homomorphism $\varphi: U \to U$ is scalar.

We'll deduce (1) from this. In Section 2 we will compute the multiplicities m_i for $A = V = \mathbb{F}G$: we'll show that

$$m_i = \dim U_i / \dim \operatorname{End}_A(U_i) \ \forall \text{ irreducibles } U_i.$$

This gives a tool to classify finite dimensional irreducible representations of a finite group G – one of the main objectives of the theory (in some cases). We'll do this for $G = S_3, S_4$.

In the last section we'll address a question: what kind of algebra can $\operatorname{End}_A(U)$ be.

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1.1) Proof of (1)

Lemma: Let U, V_1, V_2 be A-modules. Then we have a natural isomorphism $\mathrm{Hom}_A(U, V_1 \oplus V_2) \xrightarrow{\sim} \mathrm{Hom}_A(U, V_1) \oplus \mathrm{Hom}_A(U, V_2)$

Proof: Every linear map $\varphi: U \to V_1 \oplus V_2$ is of the form $\varphi(u) = (\varphi_1(u), \varphi_2(u))$ for uniquely defined linear maps $\varphi_i: U \to V_i$ (φ_i is just the i-th coordinate of φ). Using the definition of the module structure on $V_1 \oplus V_2$ – $a(v_1, v_2) = (av_1, av_2)$ – we see that φ is a homomorphism iff φ_1, φ_2 are so. So the isomorphism in the lemma is $\varphi \mapsto (\varphi_1, \varphi_2)$. $\square$

Proof of (1): Applying Lemma several times, we get:

$$\mathrm{Hom}_A(U, V) = \mathrm{Hom}_A\Big(U, \bigoplus_{i=1}^{k} U_i^{\oplus m_i}\Big) \xrightarrow{\sim} \bigoplus_{i=1}^{k} \mathrm{Hom}_A(U, U_i)^{\oplus m_i}.$$

(a) of the Schur lemma implies that when U is irreducible, $\dim \mathrm{Hom}_A(U, U_j) = 0$ if U and U_j are not isomorphic. In particular, if $U = U_i$, then $\dim \mathrm{Hom}_A(U_i, V) = m_i \dim \mathrm{Hom}_A(U_i, U_i)$. This implies (1). $\square$

Remark: Note that for A-modules U, V_1, V_2 we also have

$$\mathrm{Hom}_A(V_1\oplus V_2, U) \xrightarrow{\sim} \mathrm{Hom}_A(V_1,U)\oplus \mathrm{Hom}_A(V_2,U),\ \varphi\mapsto (\varphi|_{V_1}, \varphi|_{V_2}).$$

From here we see that $m_i = \dim \mathrm{Hom}_A(V, U_i)/\dim \mathrm{End}_A(U_i)$. Details are left as an exercise.

2) Decomposition of regular representation & applications.

2.1) Main result.

Theorem: Let G be a finite group, $\mathbb{F}$ be a field such that $\mathrm{char}\,\mathbb{F}=0$ or $\mathrm{char}\,\mathbb{F}>0$ but $\mathrm{char}\,\mathbb{F} \nmid |G|$. Then for every irreducible representation U of G, its multiplicity in $\mathbb{F}G$ is

$$m_U := \dim U/\dim \mathrm{End}_G(U).$$

Thx to Remark in Sec 1.1, Theorem follows from

Lemma: Let A be an associative algebra & V be an A-module. We have a vector space isomorphism $\mathrm{Hom}_A(A,V) \xrightarrow{\sim} V$

Proof: A map $\mathrm{Hom}_A(A,V)\to V$, $\varphi\mapsto\varphi(1)$ has inverse $V\to \mathrm{Hom}_A(A,V)$, $v\mapsto \varphi_v$ w. $\varphi_v(a)=av$. The claim that the two maps are inverse to each other is an exercise. $\square$

Corollary: There are only finitely many irreducible representations (of G) up to isomorphism. If $U_1, \dots U_k$ are all (pairwise non-isomorphic) irreducible representations, then

$$(2) \qquad |G| = \sum_{i=1}^{k} (\dim U_i)^2 / \dim \mathrm{End}_A(U_i)$$

Proof:

We can write $\mathbb{F}G = \bigoplus_{i=1}^{k} U_i^{\oplus m_i}$ (w. some m_i, perhaps, 0). Then by Thm, $m_i = \dim U_i / \dim \mathrm{End}_G(U_i) > 0$. So every irreducible occurs in $\mathbb{F}G$ w. nonzero multiplicity, hence there are finitely many of them. (2) follows by comparing the dimensions in $\mathbb{F}G = \bigoplus_{i=1}^{k} U_i^{\oplus m_i}$. $\square$

Note that (6) in the Schur lemma implies that if $\mathbb{F}$ is algebraically closed, then $\dim \mathrm{End}_G(U_i) = 1 \ \forall i$. So (2) becomes

$$(3) \qquad |G| = \sum_{i=1}^{k} (\dim U_i)^2.$$

Remark: Theorem generalizes to finite dimensional associative algebras A s.t. the regular module A is completely reducible.

2.2) Example: classification of irreducibles for S_3 & S_4.

Let $\mathbb{F}$ be algebraically closed and of char 0 (for simplicity)

1) S_3. Here we know three irreducibles: triv, sgn (1-dimensional & non-isomorphic by the first exercise in Sec 1.2 of Lec 5) & $\mathbb{F}_0^3$ (2-dimensional). We have $1^2+1^2+2^2=6=|S_3|$, so we have exhausted all irreducibles thx to (3).

2) S_4: we already know 4 different irreducibles: triv, sgn (dim 1), $\mathbb{F}_0^4$, $sgn\otimes\mathbb{F}_0^4$ (dim 3, non-isomorphic by 5) of Lemma in Sec 1.2 of Lec 5). We have $|S_4|=24$, so the sum of dimensions squared of the remaining irreducibles is $24-2(1^2+3^2)=4$.

But triv & sgn exhaust all 1-dimensional rep's of S_4. So the dimensions of the remaining irreducibles are $\geqslant 2$. We conclude that there's exactly one 2-dimensional irreducible.

Let's construct it. Recall that in S_4 we have order 4 normal subgroup $K=\{e,(12)(34),(13)(24),(14)(23)\}$. The quotient S_4/K is identified w. S_3 thx to $S_3\ltimes K=S_4$, where S_3 is embedded into S_4 as $\{\sigma \in S_4 \mid \sigma(4)=4\}$. Let V_2 be the pullback of $\mathbb{F}_0^3$ under $S_4 \to S_3$. Since $S_4 \twoheadrightarrow S_3$, an S_3-stable subspace in $\mathbb{F}_0^3$ is the same as an S_4-stable subspace in V_2, hence V_2 is irreducible.

For larger symmetric groups these easy methods won't work – and we'll need some theory to be developed later in the course.

3) Skew-fields.

Definition: An associative (unital) ring R is called a skew-field (or division ring) if any nonzero element is invertible.

Of course, every field is a skew-field. On the other hand, (a) of Schur lemma shows that if U is an irreducible A-module, then $\mathrm{End}_A(U)$ is a skew-field. In fact, any finite dimensional algebra over $\mathbb{F}$ that is a skew-field is $\mathrm{End}_A(U)$ for suitable A & U. For this, we need a definition & a lemma.

Definition: For an $\mathbb{F}$-algebra A, let A^{opp} denote the opposite algebra: the same vector space as A but with opposite product: $a \cdot^{opp} b = ba$.

Lemma: We have an algebra isomorphism $\mathrm{End}_A(A) \xrightarrow{\sim} A^{opp}$.

Proof: Let $V = A$, so we get a vector space isomorphism $A \to \mathrm{Hom}_A(A,A) \,(= \mathrm{End}_A(A))$, Lemma in Sec 2.1. It sends $b \in A$ to $\varphi_b : A \to A$, $a \mapsto ab$. Note that $\varphi_{b_1 b_2}(a) = ab_1 b_2 = \varphi_{b_2} \circ \varphi_{b_1}(a)$ (opposite order!) so that $\mathrm{End}_A(A) \xrightarrow{\sim} A^{opp}$. $\square$

Exercise: Let B be a skew-field. Then so is $A := B^{opp}$, and the regular representation of A is irreducible. Hence B indeed arises as the endomorphism algebra of an irreducible $\mathrm{End}_A(A)$.

Rem: The failure of $\mathrm{End}_A(U) \simeq \mathbb{F}$ for an irreducible A-module U is one of the main reasons why the representation theory over non-closed fields is more difficult than over closed ones.

3.1) Quaternions.

The most famous (and historically first) example of a skew-field which is not a field is the quaternions.

Definition (of $\mathbb{H}$): • Consider the following elements of $Mat_2(\mathbb{C})$:

$1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $j := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$

• Let $\mathbb{H} := \mathrm{Span}_{\mathbb{R}}(1, i, j, k)$.

Exercise: $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.

In particular, $\mathbb{H}$ is closed under multiplication and hence is an $\mathbb{R}$-subalgebra in $Mat_2(\mathbb{C})$.

Lemma: $\mathbb{H}$ is a skew-field.

Proof: For $\alpha \in \mathbb{H}$, $\alpha = a + bi + cj + dk$, set $\bar{\alpha} := a - bi - cj - dk$. Then a direct check shows $\alpha\bar{\alpha} = \bar{\alpha}\alpha = a^2 + b^2 + c^2 + d^2$. So for $\alpha \neq 0$, we have $\alpha^{-1} = \frac{1}{a^2+b^2+c^2+d^2}\bar{\alpha}$. □

$\mathbb{H}$ is the (skew-field of) quaternions.

Exercise: Show $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$ $\forall\, \alpha,\beta \in \mathbb{H}$.

Fact (to be proved much later in the course): $\mathbb{H}$ is the only noncommutative $\mathbb{R}$-algebra that is a skew-field.

For a brief article about Hamilton and his discovery of the quaternions, see Sec 4.13 in [E].

3.2) Bonus: Schur lemma for infinite dimensional representations.

The assumption in (6) of Schur's lemma that $\dim U < \infty$ is crucial. Here we investigate what happens when this condition is removed. We will be interested in sufficient condition on A so that for every irreducible A-module U we have

(Alg) Every $\varphi \in \operatorname{End}_A(U)$ is algebraic over $\mathbb{F}$.

Here and below U is an arbitrary irreducible A-module.

Problem 1: $\dim A < \infty \Rightarrow \dim U < \infty \Rightarrow$ (Alg)

Problem 2: Show that if $\mathbb{F}$ is algebraically closed, then $(Alg) \Rightarrow (6)$ of Theorem.

Problem 3: Suppose that A is commutative & finitely generated. Prove that the following claims are equivalent:

(1) "weak Nullstellensatz" (every quotient of A by a maximal ideal is a finite field extension over $\mathbb{F}$).

(2) Every irreducible representation of A is fin. dim'l.

One interesting example of an infinite dimensional associative algebra is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ (see Bonus to Lec 3).

Fact (Quillen's lemma): Let $\dim \mathfrak{g} < \infty$. Then every irreducible representation of $\mathcal{U}(\mathfrak{g})$ satisfies (Alg)

Finally, one can prove (Alg) for all finitely generated associative algebras over <u>uncountable</u> algebraically closed fields by an argument similar to the "quick and dirty" proof of the weak Nullstellensatz.

Problem 4: Let $\mathbb{F}$ be an uncountable field, A be a finitely generated associative algebra over $\mathbb{F}$ and U be an irreducible A-module. Prove the following:

(a) $\dim_{\mathbb{F}} A$ is at most countable.

(b) $U = Av$ $\forall\, v \neq 0$. Deduce that $\dim_{\mathbb{F}} U$ is at most countable.

(c) An endomorphism $\varphi \in \mathrm{End}_A(U)$ is uniquely recovered from $\varphi(v)$. Deduce that $\dim_{\mathbb{F}} \mathrm{End}_A(U)$ is at most countable.

(d) Suppose $\varphi \in \mathrm{End}_A(U)$ has no eigenvectors. Prove that the elements $(\varphi - a\, \mathrm{Id}_V)^{-1}$ for $a \in \mathbb{F}$ are linearly independent and arrive at a contradiction.