Lecture 8: Irreducible & completely reducible representations, pt 4 / Characters, pt 1

1) More applications of the Schur lemma.

2) Characters

Refs: Secs 11.2, 11.4 in [V], Secs 2.3, 4.2 in [E].

1) More applications of the Schur lemma.

Recall that part (6) of the Schur lemma (Thm in Sec 2 of Lec 6) states the following:

(*) Let U be an irreducible finite dimensional representation of an associative algebra A. Assume $\mathbb{F}$ is algebraically closed. Then every A-module endomorphism of U is scalar.

The goal of this section is to derive some consequences of (*).

1.1) The center

Definition: Let A be an associative algebra. Its center, $Z(A)$ is $\{z\in A \mid za=az, \forall a\in A\}$.

Exercise: The center is a subalgebra.

Example/exercise: Show $Z(\mathrm{Mat}_n(\mathbb{F}))=\{\text{scalar matrices}\}$.

Example: For future applications, it's important to understand the center of the group algebra $\mathbb{F}G$ (for G finite). Since elements of G form a basis of $\mathbb{F}G$ we have:

$$z\in Z(\mathbb{F}G) \Leftrightarrow gz=zg \ \forall g\in G \Leftrightarrow gzg^{-1}=z \ \forall g\in G.$$

We can write z as $\sum_{h\in G} a_h h$ and the final condition becomes $a_h=a_{g^{-1}hg}$ $\forall g,h\in G$. So, under the natural identification $\mathbb{F}G\simeq \mathrm{Fun}(G,\mathbb{F})$ $(g\mapsto \delta_g)$, $Z(\mathbb{F}G)$ consists of class functions (Sec 1 of Lec 3).

Exercise: $Z(\mathbb{H})=\mathbb{R}$.

1.2) Action of the center on irreducible modules.

Here's how the center is relevant for the study of representations:

Lemma: For any A-module V and any $z\in Z(A)$, the operator z_V of the action of z on V is an A-module endomorphism of V.

Proof: $az=za\ \forall a \Rightarrow a_V z_V = z_V a_V\ \forall a\in A \Leftrightarrow z_V: V\to V$ is an A-module homomorphism. $\square$

Combining w. (*) (a.k.a. 6) of Thm) w. the lemma we get:

Corollary: If $\mathbb{F}$ is algebraically closed & U is a finite dimensional irreducible A-module then any element of $Z(A)$ acts on U by a scalar operator.

This fact for $A=\mathbb{F}G$ will play an important role in our study of characters in the next lecture.

1.3) Irreducible representations of commutative algebras.

Corollary: Every finite dimensional irreducible representation of a commutative associative algebra A or an abelian group G is 1-dimensional.

Proof: If A is commutative, $Z(A)=A$. By the previous corollary, A acts on its fin. dim. irreducible representation V by scalars, so any subspace is a subrepresentation. This forces $\dim V=1$.

And if G is abelian, then $\mathbb{F}G$ is commutative, so we are done by the case of associative algebras. $\square$

2) Characters.

Let G be a group, $\mathbb{F}$ be a field, and V be a finite dimensional representation of G over $\mathbb{F}$. Recall (Sec 3 of Lec 1) that by the character of V we mean the function $\chi_V: G \longrightarrow \mathbb{F}$ given by

$$\chi_V(g) = tr(g_V)$$

Also recall from Sec 3 of Lec 1 that

$$\chi_V(ghg^{-1}) = \chi_V(h) \quad \forall g, h \in G.$$

So χ_V is constant on conjugacy classes. Such functions are called class functions. Their space a subspace in $\mathrm{Fun}(G, \mathbb{F})$, is denoted by $Cl(G)$.

We are going to study characters in detail as they are for crucial importance in Representation theory. But first we need to discuss what information they carry. We will see that the following holds:

Fact: Let $\operatorname{char} \mathbb{F} = 0$. Knowing χ_V is equivalent to knowing g_V up to (individual as opposed to simultaneous) conjugacy $\forall g \in G$.

We will prove Fact when $\mathbb{F}$ is algebraically closed. The general case is handled by changing the base $\mathbb{F}$ to the algebraic closure. Fact will be proved later.

Suppose $\varphi : U \to V$ is an isomorphism of representations. Then $\chi_V(g) = tr(g_V) = [g_V = \varphi \circ g_U \circ \varphi^{-1}] = tr(\varphi \circ g_U \circ \varphi^{-1}) = tr(g_U) = \chi_U(g)$.
Conversely, we will see later that if $\mathbb{F}$ is (algebraically closed – this assumption can be removed) of char 0, then $\chi_U = \chi_V$ implies that U & V are isomorphic.

2.1) Fun example of computation

Let X be a finite set acted on by G. For $g \in G$ we write $X^g := \{x \in X \mid gx = x\}$. Recall that the space $Fun(X, \mathbb{F})$ of all functions $X \to \mathbb{F}$ is a representation of G via:

$$[g.f](x) = f(g^{-1}x)$$

See Sec 5 of Lec 1.

Lemma: We have $\chi_{Fun(X,\mathbb{F})}(g) = |X^g|$

Proof:

Recall (Sec 1 of Lec 2) that we have a basis δ_x of $Fun(X, \mathbb{F})$ w. $g.\delta_x = \delta_{gx}$. In this basis, $g_{Fun(X,\mathbb{F})}$ is given by the matrix, $M(g)$, whose diagonal entries are:

$$M(g)_{xx} = \begin{cases} 1, & gx = x \\ 0, & gx \neq x \end{cases}$$

The trace $\operatorname{tr} M(g) = \sum_{x \in X} M(g)_{xx}$ is $|X^g|$ $\square$

Example: Let $X = G$ w. action by left multiplications, so that $\operatorname{Fun}(X, \mathbb{F})$ is the regular representation $\mathbb{F}G$. Then $X^g = X$ if $g = e$, and empty else. It follows that

$$\chi_{\mathbb{F}G}(g) = \begin{cases} |G|, & g = e \\ 0, & \text{else.} \end{cases}$$

2.2) Characters of irreducible representations of S_3, S_4.

The following lemma, in a sense, reduces the study of characters to the case of irreducible representations.

Lemma: Let V be a finite dimensional representation of a group G, and U be a subrepresentation. Then

$$\chi_V(g) = \chi_U(g) + \chi_{V/U}(g).$$

Proof: By Remarks in Secs 2.2 & 2.3 of Lec 2, we can find a basis in V s.t. the matrix of g_V is of the form $\begin{pmatrix} A_g & B_g \\ 0 & D_g \end{pmatrix}$, where A_g (resp. D_g) are matrices of g_U (resp. $g_{V/U}$). Then $\chi_V(g) = tr\, A_g + tr\, D_g = \chi_U(g) + \chi_{V/U}(g)$ $\square$

Example: Assume char $\mathbb{F}=0$ (for simplicity). Then, by Lemma in Sec 1.2 of Lec 5, $\mathbb{F}^n (= Fun(\{1,2,\dots n\}, \mathbb{F})) = \mathbb{F}^n_0 \oplus \mathbb{F}^n_{const}$. So $\chi_{\mathbb{F}^n_0}(g) = \chi_{\mathbb{F}^n}(g) - \chi_{\mathbb{F}^n_{const}}(g) = [\chi_{\mathbb{F}^n_{const}} = 1] = |\{i \in \{1,2,..n\} | g(i) = i\}| - 1$

Note that for the (1-dimensional) sign representation, have $\chi_{sgn}(g) = sgn(g)$. This already allows us to compute the irreducible characters (the shorthand for "characters of irreducible representations") for $G = S_3$. We present the result as the table of values on conjugacy classes labelled by the lengthes of cycles in the cycle decomposition. This collection (a partition of 3) uniquely recovers a conjugacy class.

	$1+1+1$: (e)	$2+1$ $((12))$	3 $((123))$
triv	1	1	1
$\mathbb{F}_0^3$	2	0	-1
sgn	1	-1	1

$\longleftarrow$ Example

Now proceed to $G=S_4$. The irreducible representations are (see Sec 2.2 of Lec 7) are triv, sgn, $\mathbb{F}_0^4$, $\text{sgn}\otimes\mathbb{F}_0^4$ & V_2 (the pullback of $\mathbb{F}_0^3$ under $S_4 \twoheadrightarrow S_3$). By the proof of 4) of Lemma in Sec 1.2 of Lec 5, we get that after identifying $\mathbb{F}_0^4$, $\text{sgn}\otimes\mathbb{F}_0^4$ as vector spaces, we have $g_{\text{sgn}\otimes\mathbb{F}_0^4} = \text{sgn}(g)\, g_{\mathbb{F}_0^4} \Rightarrow \chi_{\text{sgn}\otimes\mathbb{F}_0^4}(g) = \text{sgn}(g)\chi_{\mathbb{F}_0^4}(g)$. This gives the following character table:

	$1+1+1+1$	$2+1+1$	$2+2$	$3+1$	4
triv	1	1	1	1	1
$\mathbb{F}_0^4$	3	1	-1	0	-1
V_2	2	0	2	-1	0
$\text{sgn}\otimes\mathbb{F}_0^4$	3	-1	-1	0	1
sgn	1	-1	1	1	-1

To see that the character values for V_2 are as specified note that the restriction to $S_3 \subset S_4$ is $\mathbb{F}_0^3$, which gives columns 1, 2, 4. The element $(12)(34)$ acts by the identity, while $[(12)(34)](23) = (2413)$ acts as a permutation.