Lecture 9: Characters, pt. 2

1) Orthogonality of characters (w. addendum from 2/16)

Ref: Sec 11.4 in [V], Sec 4.5 in [E].

1.1) Main result.

Let $\mathbb{F}$ be an algebraically closed field, G be a finite group s.t. char $\mathbb{F} \nmid |G|$ (so by Maschke's thm, every representation of G is completely reducible). We consider the space of class functions

$$Cl(G) = \{f: G \to \mathbb{F} \mid f(ghg^{-1}) = f(h)\ \forall g, h \in G\}$$

On this space we introduce a bilinear form $Cl(G) \times Cl(G) \to \mathbb{F}$.

$$(f_1, f_2) := \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1})$$

It's symmetric: $(f_2, f_1) = \frac{1}{|G|} \sum_{g \in G} f_2(g) f_1(g^{-1}) = [\text{reorder } g \leftrightarrow g^{-1}]$

$$= \frac{1}{|G|} \sum_{g \in G} f_2(g^{-1}) f_1(g) = (f_1, f_2)$$

To a finite dimensional representation V of G we assign its character $\chi_V \in Cl(G)$, $\chi_V(g) := tr(g_V)$, Sec 2 in Lec 8.

The following is a very important basic result.

Thm: The characters of irreducible representations form an orthonormal (orthogonal w. squares = 1) basis in $Cl(G)$.

We will prove this theorem in this lecture.

Exercise: Verify the conclusion of the theorem for $G = S_3, S_4$ using the character tables in Sec 2.2

1.2) Strategy of proof.

The proof is in two big steps.

Claim 1: The characters of irreducibles span the vector space $Cl(G)$.

Claim 2: The characters of irreducibles are an orthonormal collection: $(\chi_U, \chi_V) = 1$ if $U \simeq V$ & 0 else.

Claim 2 $\Rightarrow$ these characters are linearly independent hence form a basis thx to Claim 1, thus proving the theorem.

Remark: We will only prove Claim 1 for char $\mathbb{F}=0$ in this lecture. The case of char $\mathbb{F}>0$ will be handled in the 2nd part of the class.

1.3) Proof of Claim 1

Assumption: the span of characters is not the whole $Cl(G)$.

We'll show that there's $z \in Z(\mathbb{F}G)$ (the center, Sec 1.1 in Lec 8) $z \neq 0$, that acts by 0 on all irreducibles hence on all representations. We'll apply this to the regular representation & arrive at a contradiction.

To start with, $\forall$ fin. dim. representation U of G (equiv. $\mathbb{F}G$) we can extend χ_U to $\mathbb{F}G$: $\chi_U(\alpha) = tr(\alpha_U)$ for $\alpha \in \mathbb{F}G$.

Recall, Corollary in Sec 1.2 of Lec 8, that z acts by a scalar on any irreducible representation U.

Lemma: This scalar is $\chi_U(z)/\dim U$.

Proof:

We get $z_U = a \cdot Id_U$ for some $a \in \mathbb{F} \Rightarrow tr(z_U) = a\, tr(Id_U) = a \cdot \dim U \Rightarrow a = \chi_U(z)/\dim U$. $\square$

Remark: We use $\operatorname{char} \mathbb{F} = 0$ when we divide by $\dim U$: we need to make sure that $\dim U \neq 0$ in $\mathbb{F}$.

Proof of Claim 1:

Step 1: $\dim Z(\mathbb{F}G) = \dim Cl(G) = \#$ of conjugacy classes in G.

Indeed, by Example in Sec 1.2 of Lec 8, $Z(\mathbb{F}G) = \{\sum_{h \in G} a_h h \mid a_h = a_{ghg^{-1}}\ \forall g, h\} \rightsquigarrow \dim Z(\mathbb{F}G) = \#$ conj. classes. Similarly, $\dim Cl(G)$ equals the same.

Step 2 (Assumption $\Rightarrow \exists\, z \in Z(\mathbb{F}G) \setminus \{0\}$, w. $z_U = 0\ \forall$ irreducible U).

Consider the linear equations $\chi_U(z) = 0$, where U runs over the (isomorphism classes of) irreducibles. By Assumption, this system of linear equations is equivalent to one w. $< \dim Cl(G) = [\text{Step 1}] = \dim Z(\mathbb{F}G)$ equations. So $\exists\, z \in Z(\mathbb{F}G) \setminus \{0\}$ w.

$\chi_U(z)=0$ $\forall$ irreducible U. Then $z_U=0$ by Lemma.

Step 3 ($z_V=0$ $\forall$ representation V): By Maschke's Thm (Lec 6) V is completely reducible, so (Corollary in Sec 2.1 of Lec 5) $V\simeq \bigoplus_{i=1}^{k} U_i^{\oplus m_i}$, where $U_1,\dots U_k$ are irreducible. By Step 2, $z_{U_i}=0$, hence $z_V=\operatorname{diag}(\overbrace{z_{U_1},\dots z_{U_1}}^{m_1},\dots,z_{U_k})=0$.

Step 4 (set $V:=\mathbb{F}G$ & get contradiction): $0=z_{\mathbb{F}G}\cdot 1=z$. Contradiction w. $z\neq 0$ $\square$

1.4) Strategy of proof of Claim 2

Let's explain how we prove that the characters of irreducibles form an orthonormal collection. This is based on

Theorem: Under the assumptions of Sec 1.1, we have

$$\dim \operatorname{Hom}_G(U,V)=(\chi_V,\chi_U)$$

Schur's lemma implies that, for irreducible U,V, the l.h.s is 0 if U,V are not isomorphic, and 1 if they are proving Claim 2.

Our proof of the previous theorem is based on two auxiliary results of independent interest.

Recall, Sec 2.2 of Lec 4, that if U, V are representations of G, then $\mathrm{Hom}(U,V)$ becomes a representation:

$$g.\varphi := g_V \circ \varphi \circ g_U^{-1},\ g \in G,\ \varphi \in \mathrm{Hom}(U,V).$$

Proposition 1: $\chi_{\mathrm{Hom}(U,V)}(g) = \chi_V(g)\chi_U(g^{-1})$

The 2nd result concerns the following: for a finite dimensional representation W relate $\dim W^G$ (the subspace of invariants) to χ_W:

Proposition 2: $\dim W^G = \frac{1}{|G|} \sum_{g \in G} \chi_W(g)$ (in $\mathbb{F}$).

Proof of Theorem modulo Propositions 1 & 2:

Recall (Sec 2.2 of Lec 4) that $\mathrm{Hom}_G(U,V) = \mathrm{Hom}(U,V)^G$. So $\dim \mathrm{Hom}_G(U,V) = \dim \mathrm{Hom}(U,V)^G = [\text{Prop } 2] = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(U,V)}(g)$
$= [\text{Prop. } 1] = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\chi_U(g^{-1}) = (\chi_V, \chi_U)$. $\square$

So it remains to prove Propositions 1&2.

1.5) Proof of Proposition 1

Pick $g\in G$ and let H be the (cyclic) subgroup of G generated by g. Since $|H|$ divides $|G|$, the representations of H over $\mathbb{F}$ are still completely reducible. By Prob. 1 in HW1 (or Sec 1.3 in Lec 7), the irreducible representations of H are 1-dimensional $\Rightarrow U, V$ are direct sums of 1-dimensional representations of H, and hence g_U, g_V are diagonalizable: let $u_1,\dots u_m\in U$, $v_1,\dots v_n\in V$ be eigenbases. Let $\mathrm{diag}(a_1,\dots a_m)$, $\mathrm{diag}(b_1,\dots,b_n)$ be the matrices of g_V, g_U in these bases. In particular, $\chi_V(g)=\sum_{i=1}^m a_i$, $\chi_U(g^{-1})=\sum_{j=1}^n b_j^{-1}$.

For the matrix unit $E_{ij}\in \mathrm{Hom}(U,V)$ $(i=1,\dots,m,\ j=1,\dots,n)$ we get

$$g.E_{ij}=g_V\circ E_{ij}\circ g_U^{-1}=a_ib_j^{-1}E_{ij}$$

$$\Downarrow$$

$$\chi_{\mathrm{Hom}(U,V)}(g)=tr(g_{\mathrm{Hom}(U,V)})=\sum_{i=1}^m\sum_{j=1}^n a_ib_j^{-1}=\Big(\sum_{i=1}^m a_i\Big)\Big(\sum_{j=1}^n b_j^{-1}\Big)=\chi_V(g)\chi_U(g^{-1}).$$

This finishes the proof. $\square$

Remarks: 1) In particular, $\chi_{U^*}(g) = [V \text{ is trivial}] = \chi_U(g^{-1})$. Also, recall that by Sec 2.2 of Lec 4, we have an isomorphism of representations $\mathrm{Hom}(U,V) \simeq U^* \otimes V \Rightarrow \mathrm{Hom}(U^*,V) \xrightarrow[\sim]{U^{**} \simeq U} U \otimes V$. So $\chi_{U\otimes V}(g) = \chi_V(g)\chi_{U^*}(g^{-1}) = \chi_V(g)\chi_U(g)$: the character of tensor product is the product of characters.

2) The formulas $\chi_{U^*}(g) = \chi_U(g^{-1})$ & $\chi_{U\otimes V}(g) = \chi_U(g)\chi_V(g)$ hold w/o the assumption that $\operatorname{char}\mathbb{F} \nmid |G|$. The proof is harder: one needs to deal w. generalized eigenspaces.

1.6) Proof of Proposition 2

Recall the averaging idempotent $\varepsilon = \frac{1}{|G|}\sum_{g\in G} g \in \mathbb{F}G$ (Sec 1.1 of Lec 6). The operator $\varepsilon_W: W \to W$ satisfies:

- $\operatorname{im} \varepsilon_W \subset W^G$
- $\varepsilon_W|_{W^G} = Id$

see Lemma in Sec 1.1 of Lec 6. In other words, ε_W is a projector to W^G (see Sec 1.2 of Lec 6).

What we need to prove is that

$$\dim W^G = \frac{1}{|G|} \sum_{g\in G} tr(g_W) = tr\left(\frac{1}{|G|} \sum g_W\right) = tr(\varepsilon_W)$$

Since ε_W is a projector to W^G, we reduce Proposition 2 to:

Lemma: Let W be a finite dimensional vector space, and $W_0 \subset W$ be a subspace. Let P be a projector to W_0. Then $\dim W_0 = tr\, P$.

Proof: Recall, Sec 1.2 of Lec 6, that $W = W_0 \oplus \ker P$. Both W_0 & $\ker P$ are P-stable, so $tr\, P = tr\, P|_{W_0} + tr\, P|_{\ker P}$ $= tr\, Id_{W_0} + tr\, 0 = \dim W_0$ $\square$

Addendum: alternative proof of Proposition 1

Proposition 1': Let U, V be finite dimensional representations of G over any field $\mathbb{F}$. Then

(1) $\chi_{U\otimes V}(g) = \chi_U(g)\chi_V(g)$

(2) $\chi_{U^*}(g) = \chi_U(g^{-1})$

(3) $\chi_{Hom(U,V)}(g) = \chi_V(g)\chi_U(g^{-1})$.

Proof: (1) Pick bases $u_1, \dots u_m \in U$, $v_1, \dots, v_n \in V$. Then the vectors $u_i \otimes v_j$ form a basis in $U \otimes V$. Fix $g \in G$. Let $A = (a_{ii'}) \in \mathrm{Mat}_m(\mathbb{F})$, $B = (b_{jj'}) \in \mathrm{Mat}_n(\mathbb{F})$ be the matrices of g_U, g_V. Recall that $g_{U\otimes V}(u \otimes v) = (g_U u) \otimes (g_V v)$. It follows that the coefficient of $u_i \otimes v_j$ in $g_{U\otimes V}(u_i \otimes v_j)$ is $a_{ii} b_{jj}$. So

$$\chi_{U\otimes V}(g) = tr(g_{U\otimes V}) = [\text{the sum of the } mn \text{ diagonal entries}]$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ii} b_{jj} = \left(\sum_{i=1}^{m} a_{ii}\right)\left(\sum_{j=1}^{n} b_{jj}\right) = \chi_U(g)\chi_V(g).$$

(2): Is similar in spirit. Let $\alpha_1, \dots \alpha_m$ be the dual (to $u_1, \dots u_m$) basis of U^*. Then the matrix of g_{U^*} in the basis $\alpha_1, \dots, \alpha_m$ is $(A^{-1})^T$ (transpose comes from $g_{U^*}\alpha = \alpha \circ g_U^{-1}$, composition on the right) And $\chi_{U^*}(g) = tr((A^{-1})^T) = tr(A^{-1}) = \chi_U(g^{-1})$

(3): follows from (1) & (2) & isomorphism of representations
$$\mathrm{Hom}(U,V) \xrightarrow{\sim} U^* \otimes V. \qquad \square$$