Lecture A1: Around reflection group, 1

0) What is this about?

1) Reflection groups

2) Regular polytopes.

Refs: [B], Chapters 4 & 5: for Sec 1

[C] for Sec 2

0) What is this about?

In this course, we have considered (and will consider) a bunch of finite groups: symmetric groups, (binary) dihedral groups, the binary tetrahedral groups with some more to follow.

These groups have some shared significance: they have to do with reflection groups, root systems and such.

This series of four lectures talks about these objects.

1) Reflection groups

1.1) Definition and examples

Let \( V \) be a finite dimensional vector space over \( \mathbb{R} \) equipped...
w. a scalar product. So we can consider its orthogonal group, $O(V)$.

**Definition:**
- By a reflection in $O(V)$ we mean the orthogonal reflection about a hyperplane, equivalently, an element $s \in O(V)$ w. $\ker(s - id_V) = 1$ (so that $\ker(s - id_V)$ is that hyperplane).
- By a reflection group in $O(V)$ we mean a finite subgroup generated by reflections.

**Examples:**
1) The dihedral group of order $2n$, i.e. the group of isometries of the regular $n$-gon in a $2$-dimensional space $V$. This is denoted by $I_2(n)$.

2) Consider the space $V = \mathbb{R}^n$ w. the standard scalar product. The $G = S_n$ acting on $\mathbb{R}^n$ via its permutation representation is a reflection group: a transposition $(ij)$ acts as the orthogonal reflection about the hyperplane $x_i = x_j$.

Note that the line $\{(x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$ is its orthogonal complement. Note that
$S_n \leftrightarrow O(\mathbb{R}^n)$ and is also a reflection group there. Note that $\mathbb{R}^n$ is an irreducible as a representation of $S_n$, it's called the reflection representation. The reflection group $S_n$ acting on $\mathbb{R}^n$ is often said to be of type $A_n$, ($n-1 = \dim \mathbb{R}^n$).

3) Our vector space is still $V = \mathbb{R}^n$ and we consider the group of "signed permutations": transformations that send $(x_1, \ldots, x_n)$ to $(\pm x_1, \pm x_2, \ldots, \pm x_n)$ for an arbitrary choice of signs. This group is isomorphic to $S_n \times (\mathbb{Z}/2\mathbb{Z})^n$. It is generated by reflections: about the hyperplanes $x_i = \pm x_j$ & $x_i = 0$. It's said to be of type $B_n$ (or $BC_n$), the reason for the notation will be explained in the next part.

4) We can consider the subgroup of all elements in the group of type $B_n$ that only change even number of signs. It's generated by the reflections about the hyperplanes of the form $x_i = \pm x_j$. It is said to be of type $D_n$. 

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1.2) Classification

A basic question is how to classify reflection groups $G \leq O(V)$ (up to equivalence: two pairs $G_1 \leq O(V_1), G_2 \leq O(V_2)$ are equivalent if $\exists$ a linear isometry $\varphi : V_1 \to V_2$ s.t. $G_2 = \varphi G_1 \varphi^{-1}$). One can reduce to the case when $V$ is irreducible over $\mathbb{C}$: if $V = V_1 \oplus V_2$, the direct sum of spaces w Euclidian scalar product s.t. both $V_1 \& V_2$ are $G$-stable, then there are reflection groups $G_i \leq O(V_i), i = 1, 2$, s.t. $G = G_1 \oplus G_2$ meaning that $G$ consists of transformations $\text{diag}(g_1, g_2) \in \text{End}(V), g_i \in G_i$.

If $V$ is irreducible over $\mathbb{C}$, then we say that $G$ is an irreducible reflection group.

The crucial step in the classification is the notion of a chamber. By a reflection hyperplane for $G$ we mean a hyperplane $H \subset V$ s.t. the reflection about $H$ is in $G$. A chamber in $V$ is the closure of a connected component of $V \setminus UH$, where the union is taken over all reflection hyperplanes. Here are examples of chambers.
Examples:

(I) Type $A_n$: the chambers are labelled by permutations and look like $\{(x_1,\ldots,x_n) | x_{g(1)} \geq x_{g(2)} \geq \cdots \geq x_{g(n)} \}^3$ for $g \in S_n$. An example is $\{(x_1,\ldots,x_n) | x_1 \geq x_2 \geq \cdots \geq x_n \}^3$.

(II) Type $B_n$: the chambers are labelled by signed permutations and look like $\{(x_1,\ldots,x_n) | x_1 \geq x_2 \geq \cdots \geq x_n \}^3$. An example is $\{(x_1,\ldots,x_n) | x_1 \geq x_2 \geq \cdots \geq x_n \}^3$.

Here are general facts about chambers:

Fact 1: $G$ permutes the chambers simply transitively.

Fact 2: Let $C$ be a chamber. Then every orbit of $G$ intersects $C$ at a single point.

In the examples above, these properties are immediate to check.

Exercise: Describe the chambers for the reflection groups of type $D_n$ and check Facts 1 & 2.
By a wall of a chamber $C$ we mean a reflection hyperplane $H$ s.t. $\dim (CNH) = \dim V - 1$.

Examples: In Example I, the walls of the chamber

$C = \{(x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i = 0 \& x_1 \geq \ldots \geq x_n \geq 0 \}$ are $x_i = x_{i+1}$ for $i = 1, \ldots, n-1$.

In Example II, the walls of the chamber $C = \{(x_1, \ldots, x_n) \mid x_1 \geq x_2 \geq \ldots \geq x_n \geq 0 \}$ are $x_i = x_{i+1}$, $i = 1, \ldots, n-1$, and $x_n = 0$.

Fact 3: If $G$ is irreducible, then each chamber has exactly $\dim V$ walls.

Now from $G$ we produce an unoriented multi-graph called the Coxeter diagram. Its vertices are walls. We connect two vertices, $H, H'$, with an edge if the angle between $H$ and $H'$ is $< \frac{\pi}{k}$. If the angle is $\frac{\pi}{k}$ with $k > 3$, we put $k$ as decoration on the edge. We note that the angle is always $\frac{\pi}{k}$, where $k$ is the order of $s_H s_{H'}$, with $s_H, s_{H'}$.
being the reflections about $H$ & $H$.

Examples: I: $\cdots \bullet \ i \ 3 \ \cdots \ n-2 \ n-1$
where $i$ corresponds to the wall $x_i = x_{i+1}$.

II: $\cdots \bullet \ i \ 3 \ \cdots \ n-2 \ n-1 \ 4 \ \cdots \ n$
where $i$ corresponds to the wall $x_i = x_{i+1}$ for $i < n$ & to $x_n = 0$
for $i = n$.

Exercise: The Coxeter diagram of type $D_n$ is

\[ \cdots \bullet \ 3 \ \cdots \ n-2 \ n-1 \ n \]

Here's the main classification results.

Thm: 1) An irreducible reflection group is uniquely determined
by its Coxeter diagram.

2) The following Coxeter diagrams can appear from irreducible
reflection groups (the index is always the dimension of $V$):

- $A_n$ ($n+1$), $B_n$ ($n+2$), $D_n$ ($n+4$), see above.
The diagrams $E_6, E_7, E_8$:

\[ \begin{array}{ccc}
E_6 & \cdots & E_7 & \cdots & E_8 \\
\vdots & ! & \vdots & ! & \vdots \\
\end{array} \]

- The diagram $F_4$:

- The diagrams $H_3, H_4$:

- The diagram $I_2(n)$ for $n \geq 5$ (corresponding to the dihedral groups ($n=3$ is $A_2$, $n=4$ is $B_2$, and $n=6$ case is known as $G_2$).

2) Regular polytopes

The regular polytopes is one source of how reflection groups arise (another source, root systems, will be considered in the next lecture).

We consider convex polytopes in a Euclidean space $V$, i.e. the convex hull of a finite subset of $V$. For a convex polytope we can consider its $k$-dimensional faces (that are assumed to be closed) as well as complete flags of faces: sequences

$F_1 \subset F_2 \subset \ldots \subset F_{n-1}$, where $F_i$ is a face of dim $i$. 
Example: For a triangle we have six complete flags that look like:

(we really need to take the closure of $F_2$, but this is hard to depict).

Definition: A polytope $P$ is called regular if for any two complete flags of faces, there's an isometry of $P$ mapping one flag to the other.

We can consider the group $\text{Iso}(P)$ of $P$: its elements are the isometries of $V$ fixing $P$. Now suppose that the center of $P$ is $0 \in V$ (so that the isometry group $\text{Iso}(P) \subset O(V)$).

Thm: $\text{Iso}(P)$ is a reflection group.

Examples: 1) dim 2. The isometry group of a regular $n$-gon is $\text{I}_2(n)$.

2) dim 3. There are five regular 3D polytopes: the tetra-
hedron, cube, octahedron, icosahedron & dodecahedron. The cube & octahedron share the same isometry group (they are "dual" to each other: to get the regular octahedron from the cube, take the convex hull of the centers of dimension $\dim V-1(=2)$ faces; the same procedure produces the cube out of the regular octahedron). The same applies to icosahedron vs dodecahedron.

The reflection groups that appear are $A_3$ (for the tetrahedron), $B_3$ (for the cube/octahedron), $H_3$ (for the icosahedron/dodecahedron).

**Sketch of proof of Thm:**

If $F, F'$ are complete flags of faces, then $\exists! \theta \in \text{Iso}(P)$ w. $\theta(F) = F'$. Now suppose that $F = (F_1 \subset F_2 \subset \ldots \subset F_n)$, $F' = (F'_1 \subset F'_2 \subset \ldots \subset F'_n)$ satisfy $F_j = F'_j$ for $j \neq i$ (w. some $i$). We claim that $\theta$ mapping $F$ to $F'$ is a reflection (the corresponding reflection hyperplane is spanned as a subspace by the centers of the faces $F_j, j \neq i$). One can then show that we can get any flag of faces from $F$ by changing one face at a time. $\Box$
The classification of regular polytopes in dim=3 is as follows. There are three families that exist in all dimensions: the regular simplex (generalizing the regular tetrahedron; its isometry group has type $A_n$, where $n$ is the dimension), the regular hypercube (generalizing the cube) and its dual (generalizing the regular octahedron). The latter two are dual to each other and their isometry groups are of type $B_n$.

In addition, in dim=4, there are three exceptional polytopes. One is self-dual w. isometry group of type $F_4$, the other two are dual to each other & have isometry group of type $H_4$.

References:

[B]: N. Bourbaki, Lie groups & Lie algebras. Ch 4-6.

[C]: H.S.M. Coxeter, Regular polytopes.