Lecture A1: Around reflection group, 1

0) What is this about?

1) Reflection groups

2) Regular polytopes.

Refs: [B], Chapters 4 & 5: for Sec 1.

[C] for Sec 2

0) What is this about?

In this course, we have considered (and will consider) a bunch of finite groups: symmetric groups, (binary) dihedral groups, the binary tetrahedral groups with some more to follow.

These groups have some shared significance: they have to do with reflection groups, root systems and such.

This series of four lectures talks about these objects.

1) Reflection groups

1.1) Definition and examples

Let V be a finite dimensional vector space over $\mathbb{R}$ equipped

w. a scalar product. So we can consider its orthogonal group, $O(V)$.

Definition: • By a reflection in $O(V)$ we mean the orthogonal reflection about a hyperplane, equivalently, an element $s \in O(V)$ w. $\operatorname{rk}(s - id_V) = 1$ (so that $\ker(s - id_V)$ is that hyperplane).

• By a reflection group in $O(V)$ we mean a finite subgroup generated by reflections.

Examples: 1) The dihedral group of order $2n$, i.e. the group of isometries of the regular n-gon in a 2-dimensional space V. This is denoted by $I_2(n)$.

2) Consider the space $V = \mathbb{R}^n$ w. the standard scalar product. The $G = S_n$ acting on $\mathbb{R}^n$ via its permutation representation is a reflection group: a transposition (ij) acts as the orthogonal reflection about the hyperplane $x_i = x_j$.

Note that the line $\{(x, \ldots, x)\} \subset \mathbb{R}^n$ is a subrepresentation & $\mathbb{R}^n_0 = \{(x_1, \ldots, x_n) \mid \sum_{i=1}^n x_i = 0\}$ is its orthogonal complement. Note that

$S_n \hookrightarrow O(\mathbb{R}^n_0)$ and is also a reflection group there. Note that $\mathbb{R}^n_0$ is an irreducible as a representation of S_n, it's called the reflection representation. The reflection group S_n acting on $\mathbb{R}^n_0$ is often said to be of type A_{n-1} ($n-1 = \dim \mathbb{R}^n_0$).

3) Our vector space is still $V = \mathbb{R}^n$ and we consider the group of "signed permutations": transformations that send $(x_1, \dots x_n)$ to $(\pm x_1, \pm x_2, \dots, \pm x_n)$ for an arbitrary choice of signs. This group is isomorphic to $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. It is generated by reflections: about the hyperplanes $x_i = \pm x_j$ & $x_i = 0$. It's said to be of type B_n (or BC_n, the reason for the notation will be explained in the next part.

4) We can consider the subgroup of all elements in the group of type B_n that only change even number of signs. It's generated by the reflections about the hyperplanes of the form $x_i = \pm x_j$. It is said to be of type D_n.

1.2) Classification

A basic question is how to classify reflection groups $G \subset O(V)$ (up to equivalence: two pairs $G_1 \subset O(V_1)$, $G_2 \subset O(V_2)$ are equivalent if $\exists$ a linear isometry $\varphi: V_1 \longrightarrow V_2$ s.t. $G_2 = \varphi G_1 \varphi^{-1}$). One can reduce to the case when V is irreducible over G: if $V = V_1 \oplus V_2$, the direct sum of spaces w. Euclidian scalar product s.t. both V_1 & V_2 are G-stable, then there are reflection groups $G_i \subset O(V_i)$, $i = 1,2$, s.t. $G = G_1 \oplus G_2$ meaning that G consists of transformations $\mathrm{diag}(g_1, g_2) \in \mathrm{End}(V)$, $g_i \in G_i$.

If V is irreducible over G, then we say that G is an irreducible reflection group.

The crucial step in the classification is the notion of a chamber. By a reflection hyperplane for G we mean a hyperplane $H \subset V$ s.t. the reflection about H is in G. A chamber in V is the closure of a connected component of $V \setminus \bigcup H$, where the union is taken over all reflection hyperplanes. Here are examples of chambers.

Examples:

(I) Type A_{n-1}: the chambers are labelled by permutations are look like $\{(x_1,\dots x_n) \mid x_{\sigma(1)} \geqslant x_{\sigma(2)} \geqslant \dots \geqslant x_{\sigma(n)}\}$ for $\sigma \in S_n$.
An example is $\{(x_1,\dots x_n) \mid x_1 \geqslant x_2 \geqslant \dots \geqslant x_n\}$.

(II) Type B_n: the chambers are labelled by signed permutations and look like $\{(x_1,\dots,x_n) \mid \varepsilon_1 x_{\sigma(1)} \geqslant \varepsilon_2 x_{\sigma(2)} \geqslant \dots \geqslant \varepsilon_n x_{\sigma(n)} \geqslant 0\}$.
An example is: $\{(x_1,\dots x_n) \mid x_1 \geqslant x_2 \geqslant \dots \geqslant x_n \geqslant 0\}$

Here are general facts about chambers:

Fact 1: G permutes the chambers simply transitively.

Fact 2: Let C be a chamber. Then every orbit of G intersects C at a single point.

In the examples above, these properties are immediate to check.

Exercise: Describe the chambers for the reflection groups of type D_n and check Facts 1 & 2.

By a wall of a chamber C we mean a reflection hyperplane H s.t. $\dim(C \cap H) = \dim V - 1$.

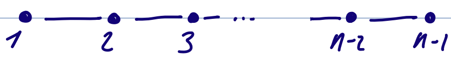
Examples: In Example I, the walls of the chamber $C = \{(x_1, \dots x_n) \mid \sum_{i=1}^{n} x_i = 0 \;\&\; x_1 \geq \dots \geq x_n\}$ are $x_i = x_{i+1}$ for $i = 1, \dots, n-1$.

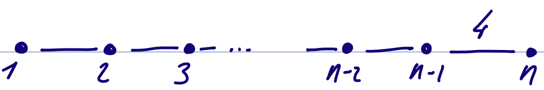
In Example II, the walls of the chamber $C = \{(x_1, \dots x_n) \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}$ are $x_i = x_{i+1}$, $i = 1, \dots, n-1$, & $x_n = 0$.

Fact 3: If G is irreducible, then each chamber has exactly $\dim V$ walls.

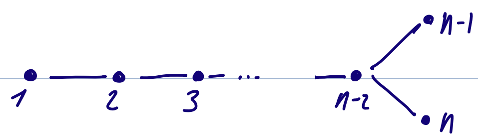
Now from G we produce an unoriented multi-graph called the Coxeter diagram. Its vertices are walls. We connect two vertices, H, H' w. an edge if the angle between H & H' is $< \frac{\pi}{2}$. If the angle is $\frac{\pi}{k}$ w. $k > 3$, we put k as decoration on the edge. We note that the angle is always $\frac{\pi}{k}$, where k is the order of $s_H s_{H'}$, with $s_H, s_{H'}$

being the reflections about H & H'

Examples: I: 
where i corresponds to the wall $x_i = x_{i+1}$.

II: 
where i corresponds to the wall $x_i = x_{i+1}$ for $i<n$ & to $x_n = 0$ for $i=n$.

Exercise: The Coxeter diagram of type D_n is



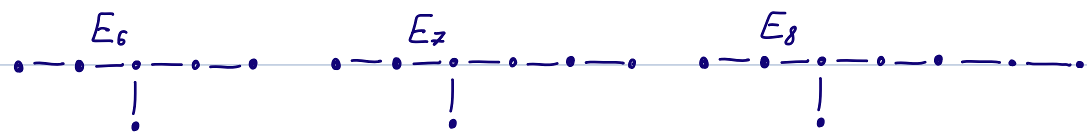
Here's the main classification results.

Thm: 1) An irreducible reflection group is uniquely determined by its Coxeter diagram.

2) The following Coxeter diagrams can appear from irreducible reflection groups (the index is always the dimension of V):

- A_n $(n \geqslant 1)$, B_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$, see above.

- The diagrams E_6, E_7, E_8:



- The diagram F_4: 

- The diagrams H_3, H_4

- The diagram $I_2(n)$ for $n \geq 5$ (corresponding to the dihedral groups ($n=3$ is A_2, $n=4$ is B_2; and $n=6$ case is known as G_2).

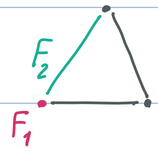
2) Regular polytopes

The regular polytopes is one source of how reflection groups arise (another source, root systems, will be considered in the next lecture).

We consider convex polytopes in a Euclidian space V, i.e. the convex hull of a finite subset of V. For a convex polytope we can consider its k-dimensional faces (that are assumed to be closed) as well as complete flags of faces: sequences

$F_1 \subset F_2 \subset \ldots \subset F_{n-1}$, where F_i is a face of $\dim = i$.

Example: For a triangle we have six complete flags that look like:



(we really need to take the closure of F_2, but this is hard to depict).

Definition: A polytope P is called regular if for any two complete flags of faces, there's an isometry of P mapping one flag to the other.

We can consider the group $Iso(P)$ of P: its elements are the isometries of V fixing P. Now suppose that the center of P is $0 \in V$ (so that the isometry group $Iso(P) \subset O(V)$).

Thm: $Iso(P)$ is a reflection group.

Examples: 1) dim 2. The isometry group of a regular n-gon is $I_2(n)$.

2) dim 3. There are five regular 3D polytopes: the tetra-

hedron, cube, octahedron, icosahedron & dodecahedron. The cube & octahedron share the same isometry group (they are "dual" to each other: to get the regular octahedron from the cube, take the convex hull of the centers of dimension $\dim V-1 (=2)$ faces; the same procedure produces the cube out of the regular octahedron). The same applies to isosahedron vs dodecahedron.

The reflection groups that appear are A_3 (for the tetrahedron), B_3 (for the cube/octahedron), H_3 (for the icosahedron/dodecahedron).

Sketch of proof of Thm:

If $F_\bullet, F_\bullet'$ are complete flags of faces, then $\exists!\ \theta \in Iso(P)$ w. $\theta(F_\bullet) = F_\bullet'$. Now suppose that $F_\bullet = (F_1 \subset F_2 \subset \ldots \subset F_{\ell-1})$, $F_\bullet' = (F_1' \subset F_2' \subset \ldots \subset F_{\ell-1}')$ satisfy $F_j' = F_j$ for $j \neq i$ (w. some i). We claim that θ mapping F to F' is a reflection (the corresponding reflection hyperplane is spanned as a subspace by the centers of the faces $F_j, j\neq i$). One can then show that we can get any flag of faces from F by changing one face at a time. $\square$

The classification of regular polytopes in dim >3 is as follows. There are three families that exist in all dimensions: the regular simplex (generalizing the regular tetrahedron; its isometry group has type A_n, where n is the dimension), the regular hypercube (generalizing the cube) and its dual (generalizing the regular octahedron). The latter two are dual to each other and their isometry groups are of type B_n.

In addition, in dim $=4$, there are three exceptional polytopes. One is self-dual w. isometry group of type F_4, the other two are dual to each other & have isometry group of type H_4.

References:

[B]: N. Bourbaki, Lie groups & Lie algebras. Ch. 4-6.

[C] H.S.M. Coxeter, Regular polytopes.