Lecture A2: Root systems.

0) Motivation:

In Lecture A1, we have talked about reflection groups. In turns out that many of them arise from another object of combinatorial nature: root systems. Those that arise in this way are precisely reflection groups that preserve a lattice, they are also known as Weyl groups. Root systems & Weyl groups are very important for various things in Math, for example, in the study of semisimple Lie groups & their Lie algebras. And representations of Weyl groups are important in the general Representation theory as well - but this is yet more advanced.

1) Root systems

1.1) Definition & examples.

Let V be a Euclidien space w. scalar product $(\cdot,\cdot)$. For

$\alpha \in V \setminus \{0\}$ consider the reflection s_α w.r.t. $\alpha^\perp$, an important exercise is that $s_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha,\alpha)}\alpha$.

Let $\Delta \subset V\setminus\{0\}$ be a finite subset.

Definition: We say that Δ is a root system if:

- $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ $\forall$ $\alpha, \beta \in \Delta$,
- $s_\alpha(\Delta) = \Delta$ $\forall$ $\alpha \in \Delta$.
- $\mathrm{Span}_{\mathbb{R}}(\Delta) = V$.

Exercise: $\Delta = -\Delta$.

Examples:

1) Type A_n: $V = \mathbb{R}_0^{n+1}$ w. scalar product restricted from the standard product on $\mathbb{R}^{n+1}$. Let $e_1, \dots e_{n+1}$ be the tautological basis in $\mathbb{R}^{n+1}$. Then $\Delta = \{e_i - e_j \mid i \neq j\}$ is a root system.

2) Type B_n: $V = \mathbb{R}^n$ ($e_1, \dots e_n$ is still the tautological basis). Then $\Delta = \{\pm e_i \pm e_j,\ 1 \leq i < j \leq n,\ \pm e_i\ (1 \leq i \leq n)\}$ is a root system.

3) Type C_n: $V=\mathbb{R}^n$. Then $\Delta = \{\pm e_i \pm e_j,\ 1\le i<j\le n,\ \pm 2e_i\ (1\le i\le n)\}$ is a root system.

4) Type D_n: $V=\mathbb{R}^n$. Then $\Delta = \{\pm e_i \pm e_j,\ 1\le i<j\le n\}$ is a root system.

5) Type E_8. Consider the following subgroup in $\mathbb{R}^8$:

$$\Gamma := \left\{ \sum_{i=1}^{8} x_i e_i \;\middle|\; \bullet \sum_{i=1}^{8} x_i \in 2\mathbb{Z} \ \& \ \bullet\, x_i \in \mathbb{Z}\ \forall i \text{ or } x_i \in \mathbb{Z}+\tfrac{1}{2}\ \forall i \right\}.$$

The point is that this subgroup is an "even" & "uni-modular" lattice, where "even" means that $(\gamma,\gamma)\in 2\mathbb{Z}$ $\forall\ \gamma\in\Gamma$ & "unimodular" means that the determinant of the Gram matrix of any basis in Γ is ± 1.

For Δ we take the subset of length 2 elements in Γ. To check that this is a root system

1.2) Weyl group.

From a root system Δ we can recover a reflection group:

take the group W generated by s_α w. $\alpha \in \Delta$. To check that it's finite is left as an exercise. The group W is known as the Weyl group associated w. Δ.

Examples: 1) Root systems of types A_n, B_n, D_n give rise to the reflection groups of those types. The root system of type C_n gives the Weyl group of type B_n.

2) For two different reflections $s_\alpha, s_\beta \in W$ we have $(s_\alpha s_\beta)^m = e$ for $m = 2, 3, 4$ or 6, this is left as an exercise, use the condition that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$. This shows that reflection groups of types $H_3, H_4, I_2(m)$ for $m \neq 2,3,4,6$ cannot appear as Weyl groups. On the other hands all other irreducible reflection groups arise as Weyl groups: one can explicitly construct the root systems corresponding to E_6, E_7, E_8 (the latter has been treated above), F_4 & G_2 a.k.a. $I_2(6)$.

An important remark is that, since W preserves Δ, it also preserves $\mathrm{Span}_{\mathbb{Z}}(\Delta)$, which is a "lattice" in V: a finitely generated abelian group that spans V. Conversely, one can show that a reflection group that preserves a lattice must be the Weyl group of some root system.

1.3) Dynkin diagrams & classification.

One can classify root systems using so called Dynkin diagrams. In order to explain the classification result we need to give two definitions.

Definition: We say that a root system Δ is:

- irreducible if Δ cannot be split as the union of two non-empty subsets $\Delta_1 \sqcup \Delta_2$ w. Δ_2 being orthogonal to Δ_1
- reduced if $\alpha \in \Delta \Rightarrow 2\alpha \notin \Delta$.

The classification of root systems reduces to that of irreducible root systems. The meaning of reduced systems is more subtle: the initial reason why one restricts to reduced root systems is that every root system arising from a semisimple Lie algebra must be reduced.

Now let Δ be an irreducible reduced root system. Choose a chamber, say C, for the Weyl group W. By the system of simple roots, $\Pi (= \Pi_C)$, associated to C, we mean the collection of roots α satisfying the following conditions:

- The pairing $w.\alpha$ is $\geqslant 0$ on C.
- $\alpha^\perp$ is one of the walls of C.

We extract a kind of graph from Π. The vertices correspond to simple roots. The number of edges (non-oriented) between α & β is $\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$. In addition, if $(\alpha,\alpha) > (\beta,\beta)$, then we put the decoration $>$ in the direction from α to β.

Example: Here are examples of simple root systems for root systems of types A, B, C, D:

- A_n: $e_i - e_{i+1}$, $i = 1, \ldots n$.
- B_n: $e_i - e_{i+1}$, $i = 1, \ldots, n-1$, e_n.

- C_n: $e_i - e_{i+1}$, $i = 1, \dots, n-1$, $2e_n$.
- D_n: $e_i - e_{i+1}$, $i = 1, \dots, n-1$, $e_{n-1} + e_n$.

Exercise: Show that the Dynkin diagrams are as follows:

A_n: $\bullet - \bullet - \bullet - \cdots \bullet - \bullet$

B_n: $\bullet - \bullet - \bullet - \cdots \bullet \Rightarrow \bullet$

C_n: $\bullet - \bullet - \bullet - \cdots \bullet \Leftarrow \bullet$

D_n: $\bullet - \bullet - \bullet - \cdots \bullet <$

Theorem: 1) An irreducible reduced root system is uniquely recovered from its Dynkin diagram.

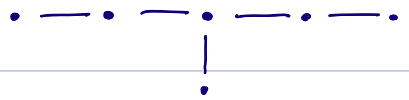
2) The following Dynkin diagrams occur:

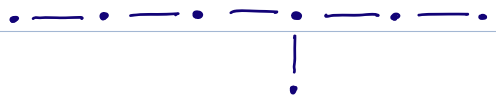
A_n: $\bullet - \bullet - \bullet - \cdots \bullet - \bullet$, $n \geq 1$

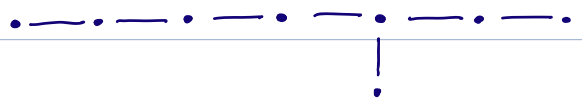
B_n: $\bullet - \bullet - \bullet - \cdots \bullet \Rightarrow \bullet$, $n \geq 2$

C_n: $\bullet - \bullet - \bullet - \cdots \bullet \Leftarrow \bullet$ $n \geq 3$

D_n: $\bullet - \bullet - \bullet - \cdots \bullet <$ $n \geq 4$

E_6: 

E_7: 

E_8: 

F_4: 

G_2