Lecture A3: complex reflection groups.

Ref: [B] (see Lec A1), Sec 5 in Ch. 5.

0) Motivation: we have discussed the reflection groups in Lec A1. These are groups acting on real vector spaces. We can complexify and view them as subgroups of $GL(V)$, where V is a finite dimensional complex vector space. One can ask whether one can describe the resulting subgroups of $GL(V)$ w/o a reference to Euclidian vector spaces and characterize them via some properties. The answer to the first question is "not quite", but the answer to the 2nd one is "yes."

1) Complex reflection groups.

Let's start w. definitions. Let V be a finite dimensional complex vector space.

Definition: • An element $s \in GL(V)$ is called a complex reflection if s has finite order & $rk(s - id_V) = 1$.

• A finite subgroup $G \subset GL(V)$ is called a complex reflection group if it is generated by complex reflections.

Examples: 1) Let $V_{\mathbb{R}}$ be a Euclidian vector space and V be its complexification. Then any reflection group in $O(V_{\mathbb{R}})$ is also a complex reflection group (in $GL(V)$)

2) Here is an infinite series of complex reflection groups. Fix integers $n, \ell \geq 1$. Consider the subgroup $\mu_\ell \subset \mathbb{C} \setminus \{0\}$ of ℓth roots of 1. Let $V = \mathbb{C}^n$. Finally, let $G(\ell,1,n) := S_n \ltimes \mu_\ell^n$, where S_n permutes the n copies of μ_ℓ. There's a natural representation of $G(\ell,1,n)$ in V: the elements of S_n permute the n copies of $\mathbb{C}$, while elements of μ_ℓ^n act by $(\xi_1, \ldots \xi_n) . (z_1, \ldots z_n)$ $= (\xi_1 z_1, \ldots, \xi_n z_n)$, $(\xi_1, \ldots \xi_n) \in \mu_\ell^n$ & $(z_1, \ldots z_n) \in V = \mathbb{C}^n$. This gives an embedding $G(\ell,1,n) \hookrightarrow GL(V)$. This is indeed a complex reflection group: it's generated by complex reflections $(i,j) \in S_n$ & $(1, \ldots, 1, \varepsilon, 1, \ldots, 1) \in \mu_\ell^n$. Note also that for $\ell = 1$, we, essentially, recover the reflection group of type A_{n-1}, while for $\ell = 2$, we get

B_n.

3) One can generalize the previous example as follows. Pick d dividing ℓ. Consider the subgroup $G(\ell,d,n)\subset G(\ell,1,n)$ consisting of all elements $\sigma(\varepsilon_1,\dots,\varepsilon_n)$ (w. $\sigma\in S_n$, $(\varepsilon_1,\dots,\varepsilon_n)\in\mu_\ell^n$) s.t.

$$(\varepsilon_1\varepsilon_2\dots\varepsilon_n)^{\ell/d}=1$$

For example, for $\ell=d=2$, we recover the reflection group of type D_n.

Exercise: $G(\ell,d,n)\subset GL_n(\mathbb{C})$ is a complex reflection group.

Fact: Apart from the series $G(\ell,d,n)$, there are only finitely many "irreducible" complex reflection groups.

4) Here's an example of an exceptional complex reflection group that doesn't come from a real reflection group. This group consisting of 24 elements (and so denoted by G_{24}) has appeared in the extra credit part of Problem 3 in HW 3.

The complex reflection groups were fully classified by Shephard & Todd in the 50's.

2) Structure of invariants (requires 380)

Here's a very nice characterization of complex reflection groups in terms of invariants. Recall, Lecture 6.5, that for a finite subgroup $G \subset GL(V)$ one can consider the subalgebra of invariants $\mathbb{C}[V]^G \subset \mathbb{C}[V]$, which turns out to be finitely generated. In particular, $\mathbb{C}[V]$ is a module over $\mathbb{C}[V]^G$.

Problem: Show that $\mathbb{C}[V]$ is integral over $\mathbb{C}[V]^G$ & deduce that it is a finitely generated module. Moreover, use Galois theory to show that the number of generators is always $\geq |G|$.

And here's a conceptual characterization of complex reflection groups in terms of invariants.

Theorem (Chevalley - Shephard - Todd) TFAE:

(a) $G \subset GL(V)$ is a complex reflection group.

(b) $\mathbb{C}[V]^G$ is isomorphic to the algebra of polynomials (automatically, in $\dim V$ variables)

(c) $\mathbb{C}[V]$ is a free module over $\mathbb{C}[V]^G$ (automatically of rank $|G|$).

A conceptual proof of this theorem is due to Chevalley & can be found in Bourbaki's book.

Example: Suppose $G = S_n$ acting on $\mathbb{C}^n$, the permutation representation. Then $G \subset GL(\mathbb{C}^n)$ is a complex reflection group. (b) follows from the fundamental theorem about symmetric polynomials. (c) is left as an exercise.

Exercise: Prove (b) for reflection groups of types B_n & D_n.