Lecture A4: symplectic reflection groups.

0) Motivation, part 1.

In this final lecture of the series we are going to discuss the class of subgroups in $GL(U)$ (for suitable U) that, in a way, generalizes complex reflection groups but also the finite subgroups of $SL_2(\mathbb{C})$ that we have seen in Homeworks 1 & 3. These are so called "symplectic reflection groups." These groups are subgroups in symplectic groups $Sp(U)$, so we will start by reviewing symplectic vector spaces and symplectic groups.

0.1) Background on symplectic vector spaces.

Let U be a finite dimensional vector space$/\mathbb{C}$. By a symplectic form on U we mean a non-degenerate skew-symmetric bilinear form $U\times U \to \mathbb{C}$. A usual notation for such a form is ω. When U is equipped w. a symplectic form, we say that it is a symplectic vector space.

Example 1: On $\mathbb{C}^2$, we have the symplectic form "det":
$\omega_1((a,b),(c,d)) = ad-bc$. More generally on $(\mathbb{C}^2)^{\oplus n}$ we can consider the direct sum of several copies of det:
$\omega((x_1,\dots x_n),(y_1,\dots y_n)) = \sum_{i=1}^{n} \omega_1(x_i,y_i)$, $x_i, y_i \in \mathbb{C}^2$. It is symplectic.

Example 2: Let V be a finite dimensional vector space. Then $U = V^* \oplus V$ carries a natural symplectic form

$$\omega((\alpha,v),(\alpha',v')) = \alpha(v') - \alpha'(v), \quad \alpha,\alpha' \in V^*, \ v,v' \in V.$$

In the previous examples the space U is always even dimensional. This holds in general. Moreover, for every symplectic form ω, there's a basis $u_1, v_1, u_2, v_2, \dots u_n, v_n \in U$ s.t. the form is given on the basis by:

$$\omega(u_i,u_j) = \omega(v_i,v_j) = 0, \quad \omega(u_i,v_j) = \delta_{ij} \ (= -\omega(v_j,u_i)).$$

Such a basis is usually called a Darboux basis.

For proofs see Sec 5.3 in [V].

Remark: One reason why symplectic structures are interesting is that they appear in Hamiltonian Mechanics.

0.2) Symplectic groups.

Let U be a symplectic vector space w. form ω. By the symplectic group, $Sp(U)$, we mean the subgroup of all elements $g \in GL(U)$ that preserve ω, i.e.

$$\omega(gu, gv) = \omega(u,v) \ \forall u,v \in U.$$

Example: Let $U = V \oplus V^*$ as in Example 2 of Sec 0.1. We claim that the subgroup of all elements of $Sp(U)$ that preserve the decomposition $U = V \oplus V^*$ is identified w. $GL(V)$. Indeed, for $g \in GL(V)$, let $g^*: V^* \to V^*$ be the induced linear map (given by $[g^*\alpha](v) = \alpha(gv)$). An element $\mathrm{diag}(g, g^{*-1}): V \oplus V^* \xrightarrow{\sim} V \oplus V^*$ preserves ω (exercise) & hence lies in $Sp(U)$. Conversely, if $h \in Sp(U)$ preserves the direct sum decomposition $V \oplus V^*$, then it is of the form $\mathrm{diag}(g, g^{*-1})$ for $g := h|_V$ (exercise).

1) Symplectic reflection groups.

1.1) Definition and basic examples.

We start w. an exercise. Let U be a symplectic vector space.

Exercise: Let $g \in Sp(U)$ be a finite order element. Then the restriction of ω to $\{u \in U \mid gu = u\}$ is nondegenerate. Hence $\dim\{u \in U \mid gu = u\}$ is even dimensional.

So $\mathrm{rk}(g - id_U) \geq 2$ if $g \neq id_U$.

Definition: A symplectic reflection in $Sp(U)$ is a finite order element g s.t. $\mathrm{rk}(g - id_U) = 2$. A finite subgroup of $Sp(U)$ is called a symplectic reflection group if it is generated by symplectic reflections.

Example 1: Let $\dim U=2$. Then every finite subgroup of $Sp(U)(\simeq SL_2(\mathbb{C}))$ is a symplectic reflection group.

Example 2: Let $G_1\subset SL_2(\mathbb{C})$ be a finite subgroup. Let $U=(\mathbb{C}^2)^{\oplus n}$ be equipped with the structure of a symplectic vector space as in Example 1 of Sec 0.1. Set $G_n:=S_n\ltimes G_1^n$. We equip U with the structure of a representation of G_n by letting S_n to permute the n summands of $\mathbb{C}^2$ and the n copies of G_1 to act on their copies of $\mathbb{C}^2$ (compare with the construction of $G(\ell,1,n)\subset GL_n(\mathbb{C})$ in Sec 1 of Lec A3). We leave it as an exercise to check that the image of G_n in $GL(U)$ lies in $Sp(U)$ and that it's generated by symplectic reflections.

Example 3: Let $G\subset GL(V)$ be a complex reflection group. Embedding $GL(V)$ into $Sp(U)$ w. $V=U\oplus U^*$ as explained in Sec 0.2, we can view G as a subgroup in $Sp(U)$. Every complex reflection in $GL(V)$ is a symplectic reflection

in $Sp(U)$. So $G \subset Sp(U)$ is a symplectic reflection group.
Note that applying this construction to $G(\ell,1,n) \subset GL(\mathbb{C}^n)$ we get the group $S_n \ltimes G_1^n \subset Sp((\mathbb{C}^2)^{\oplus n})$ w.

$$G_1 = \{ \mathrm{diag}(\varepsilon, \varepsilon^{-1}) \mid \varepsilon^\ell = 1 \}$$

Symplectic reflection groups were classified by Cohen in 1980.

1.2) Classification in dim 2.

The classification in general isn't particularly nice, but it is very nice in dimension 2, where we are concerned w. describing the finite subgroups of $SL_2(\mathbb{C})$ (up to conjugation in $SL_2(\mathbb{C})$).

Theorem: Up to conjugation in $SL_2(\mathbb{C})$, the finite subgroups in $SL_2(\mathbb{C})$ (different from $\{1\}$) are classified by Dynkin diagrams of types A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, E_8.

Examples: The cyclic group $\{\mathrm{diag}(\varepsilon, \varepsilon^{-1}) \mid \varepsilon^n = 1\}$ corresponds to the diagram A_{n-1}, while the binary dihedral group w. $4n$ elements corresponds to the diagram D_{n+2}.

Sketch of proof of Thm:

Let G denote a finite subgroup of $SL_2(\mathbb{C})$.

Step 1: Recall, Sec 1.3 of Lec 6, that there's a G-invariant hermitian scalar product on $\mathbb{C}^2$. Equivalently, G is conjugate (in $SL_2(\mathbb{C})$) to a subgroup in SU_2, the group of unitary transformations of $\mathbb{C}^2$ w. determinant 1. One can further show that if finite subgroups of SU_2 are conjugate in $SL_2(\mathbb{C})$, then they are conjugate in SU_2. So we reduce our problem to classifying finite subgroups of SU_2 up to conjugacy.

Step 2: There's a group epimorphism $SU_2 \longrightarrow SO_3(\mathbb{R})$. It's constructed as follows. Let H_2 denote the space of Hermitian 2×2-matrices, it's a 3-dimensional vector space over $\mathbb{R}$ that comes w. Euclidian scalar product: $(A,B) := tr(AB)$.

The group SU_2 acts on H_2 by $g.A = gAg^{-1}$. This action is by linear isometries preserving the scalar product. This defines a group homomorphism from SU_2 to the orthogonal group of H_2, which is identified w. $O_3(\mathbb{R})$. One can show (in the increasing order of difficulty) that:

- the kernel of this homomorphism is $\{\pm I\}$, where I is the identity.
- the image is contained in $SO_3(\mathbb{R})$ (can be deduced, say, from the spectral theorem).
- the image coincides w. $SO_3(\mathbb{R})$.

Step 3: The classification of finite subgroups of $SO_3(\mathbb{R})$ is known, see Problem 4.12.8 in [E]. The answer is as follows: these subgroups are either the image of the two families of the subgroups in Example above or groups of rotations (= symmetries preserving the orientation) of the regular polyhedra: the tetrahendron, the cube/octahedron and the icosahedron/dodecahedron.

Step 4: Now we have classified all finite subgroups of $SO_3(\mathbb{R})$. By taking the preimage we recover the classification of the finite subgroups of SU_2 containing $\pm I$. On the other hand, if $G \subset SU_2$ is a finite subgroup, then $G\{\pm I\}$ is also a finite subgroup. So, to complete the classification we need to answer the following question: when a finite subgroup $\tilde{G} \subset SU_2$ contains a subgroup $G \subset \tilde{G}$ s.t. $-I \notin G$ & $G\{\pm I\} = \tilde{G}$. One can analyze this case by case and conclude that this is only possible for $\tilde{G} = \{diag(\varepsilon, \varepsilon^{-1}) \mid \varepsilon^{2\ell} = 1\}$ w. odd ℓ & $G = \{diag(\varepsilon, \varepsilon^{-1}) \mid \varepsilon^{\ell} = 1\}$.

Step 5: It remains to assign a Dynkin diagram to each of the groups (cyclic, binary dihedral and the three exceptional groups - binary tetrahedral, octohedral and icosahedral ones). This is done using the recipe outlined in Psets 1 & 3: we form the unoriented graph whose vertices correspond to the irreducible representations of G and between vertices U, U' we have $\dim \operatorname{Hom}_G(\mathbb{C}^2 \otimes U, U')$ edges. Then we remove the vertex corresponding to the trivial representation getting a Dynkin diagram. It's a matter of computation to show that:

- The cyclic group w. n elements gives the diagram of type A_{n-1}
- The binary dihedral group w. $4n$ elements corresponds to D_{n+2}
- The binary tetrahedral, octahedral & icosahedral groups correspond to E_6, E_7, E_8, respectively. $\square$