Lecture B1: More on representations of symmetric groups, pt. 1: Alternative take on the construction

1) Idempotents & left ideals generated by them.
2) Young symmetrizers & classifications of irreducibles for S_n

Refs: [E], Secs 5.12, 5.13.

The goal of this bonus lecture is to present the same construction of irreducible representations of S_n (over an algebraically closed field of characteristic 0) but using a different language.

1) Idempotents & left ideals generated by them.

Let A be an associative algebra over $\mathbb{F}$. An element $\varepsilon \in A$ is called an idempotent if $\varepsilon^2 = \varepsilon$. For an idempotent ε we consider the left ideal $A\varepsilon \subset A$. We want to investigate $A\varepsilon$ as an A-module. First, some motivating examples.

Examples: 1) Let $A = Mat_n(S)$, where S is a skew-field. Let $\varepsilon = E_{11}$, a matrix unit. Then $A\varepsilon \simeq S^n$, the module of column vectors.

2) Let H be a finite group and U be its 1-dimensional representation. Let $\chi: H \to \mathbb{F}\setminus\{0\}$ be the corresponding group homomorphism. Suppose $\operatorname{char} \mathbb{F} \nmid |H|$.

Set $A = \mathbb{F}H$ and let $\varepsilon_U \in A$ be given by

$$\varepsilon_U := \frac{1}{|H|} \sum_{h\in H} \chi^{-1}(h)\, h.$$

It follows that $g\varepsilon_U = \chi(g)\varepsilon_U \ \forall g \in H$. In particular, $A\varepsilon \simeq U$.

2') Now suppose H is a subgroup of G, so that $\mathbb{F}H$ is a subalgebra of $\mathbb{F}G$. We can view the idempotent $\varepsilon_V \in \mathbb{F}H$ as an element of $\mathbb{F}G$. We want to describe $(\mathbb{F}G)\varepsilon_U$, a representation of G: we claim that it gets identified with $\operatorname{Ind}_H^G U$. Indeed, $\operatorname{Ind}_H^G U$, by definition, is $\{f \in \operatorname{Fun}(G,\mathbb{F}) \mid f(gh^{-1}) = \chi^{-1}(h) f(g) \ \forall g\in G, h \in H\}$. If we identify $\mathrm{Fun}(G,\mathbb{F})$ w. $\mathbb{F}G$ by sending δ_g to g, $\forall g\in G$, $\mathrm{Ind}_H^G U\subset \mathrm{Fun}(G,\mathbb{F})$ is identified w. $(\mathbb{F}G)\varepsilon$ (exercise).

Finally, we describe the space of homomorphisms from a module of the form $A\varepsilon$ to an arbitrary module.

Lemma: Let $\varepsilon\in A$ be an idempotent & M be an A-module. Then $\mathrm{Hom}_A(A\varepsilon, V)\xrightarrow{\sim}\varepsilon V$ via $\varphi\mapsto\varphi(\varepsilon)$.

Proof:

Note that $\varphi(\varepsilon)\in\varepsilon V$: $\varphi(\varepsilon)=\varphi(\varepsilon^2)=\varepsilon\varphi(\varepsilon)$. This gives a map $\mathrm{Hom}_A(A\varepsilon,V)\longrightarrow\varepsilon V$. For $u\in\varepsilon V$ define $\varphi_u: A\varepsilon\to V$ via $\varphi_u(b)=bu$. To check that these maps are mutually inverse is an exercise.

2) Young symmetrizers & classifications of irreducibles for S_n

We want to realize the irreducible representation V_λ (where λ is a partition of n) as $(\mathbb{F}S_n)\varepsilon_\lambda$ for a suitable idempotent $\varepsilon_\lambda\in\mathbb{F}S_n$ ($\mathbb{F}$ is alg. closed & char $\mathbb{F}=0$)

We fill the diagram λ w. numbers from 1 to n by putting $1, \ldots \lambda_1$ in the 1st row, $\lambda_1+1, \ldots \lambda_1+\lambda_2$ in the second row, etc. E.g $\begin{array}{|c|c|c|}\hline 6 \\ \cline{1-2} 4 & 5 \\ \hline 1 & 2 & 3 \\ \hline\end{array}$. As in Lec 16, consider the subgroup $S_\lambda \subset S_n$ consisting of all $\sigma \in S_n$ preserving the subsets of elements in the rows. Also consider the subgroup S_λ' consisting of all $\tau \in S_n$ that preserves the subsets of elements corresponding to columns, for $\lambda = (3,2,1)$, as above these subsets are $\{1,4,6\}, \{2,5\}, \{3\}$. Note that $S_\lambda \cap S_\lambda' = \{e\}$ and that S_λ' is conjugate to S_{λ^t}.

Let a_λ denote the idempotent in S_λ corresponding to the trivial representation, $a_\lambda = \frac{1}{|S_\lambda|} \sum_{\sigma \in S_\lambda} \sigma$. And let b_λ be the idempotent in S_λ' corresponding to the sign representation: $b_\lambda = \frac{1}{|S_\lambda'|} \sum_{\tau \in S'} sgn(\tau)\tau$.

In the notation from Lec 16,

$$(\mathbb{F} S_n) a_\lambda \simeq I_\lambda^+, \quad (\mathbb{F} S_n) b_\lambda \simeq I_\lambda^-.$$

Recall, Main Claim of Lec 16, that $\dim \operatorname{Hom}_{S_n}(I_\lambda^+, I_\lambda^-) = 1$. By Lemma in Sec 1, $\dim a_\lambda (\mathbb{F} S_n) b_\lambda = 1$.

Consider the element $c_\lambda = a_\lambda b_\lambda \in a_\lambda (\mathbb{F} S_n) b_\lambda$. It's $\neq 0$.

Lemma: $c_\lambda^2 = n_\lambda c_\lambda$ for $n_\lambda \in \mathbb{F} \setminus \{0\}$.

Proof: $c_\lambda^2 = a_\lambda b_\lambda a_\lambda b_\lambda \in a_\lambda (\mathbb{F}S_n) b_\lambda$. The space is 1-dimensional and $c_\lambda \neq 0$. So $c_\lambda^2 = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{F}$. We need to show $n_\lambda \neq 0$. Assume the contrary: $c_\lambda^2 = 0$. Then $\chi_V(c_\lambda) = 0$ for all representations V of $\mathbb{F}S_n$ including $V = \mathbb{F}S_n$. Recall that $\chi_{\mathbb{F}S_n}(g) = \delta_{g,e}$. If $\chi_{\mathbb{F}S_n}(c_\lambda) = 0$, then the coefficient of e in c_λ is zero. But from $S_\lambda \cap S_\lambda' = \{e\}$, it's easy to see that the coefficient of e in c_λ is $\frac{1}{|S_\lambda||S_\lambda'|}$. This contradiction finishes the proof. $\square$

So $\varepsilon_\lambda := n_\lambda^{-1} c_\lambda$ is an idempotent. It's called the Young symmetrizer.

Proposition: $(\mathbb{F}S_n)\varepsilon_\lambda \xrightarrow{\sim} V_\lambda$

Proof: Observe that $(\mathbb{F}S_n)\varepsilon_\lambda$ occurs as a subrepresentation in I_λ^-: $(\mathbb{F}S_n)\varepsilon_\lambda = (\mathbb{F}S_n) a_\lambda b_\lambda \subset (\mathbb{F}S_n) b_\lambda$. Next, observe that $v \mapsto v b_\lambda$ defines a homomorphism

$$I_\lambda^+ = (\mathbb{F}S_n)a_\lambda \longrightarrow (\mathbb{F}S_n)b_\lambda = I_\lambda^-$$

whose image is $(\mathbb{F}S_n)\varepsilon_\lambda$. Since V_λ is the only irreducible that occurs both in I_λ^+, I_λ^-, the image of any nonzero homomorphism $I_\lambda^+ \to I_\lambda^-$ is V_λ. Proposition follows. $\square$

Example: Two easy examples of Young symmetrizers are $\varepsilon_\lambda = a_\lambda$ for $\lambda = (n)$ & $\varepsilon_\lambda = b_\lambda$ for $\lambda = (1, \ldots 1)$.