Lecture 21: More on representations of symmetric groups, pt. 2
Schur-Weyl duality.

1) Schur-Weyl duality.

Refs: [E], Secs 5.18, 5.19.

1.0) Introduction.

One reason why the symmetric groups are important for Representation theory is that their representations are closely related to representations of other groups/algebras. The most classical of these connections is the Schur-Weyl duality (discovered by Schur) that relates the representations of symmetric groups and of general linear groups.

1.1) Polynomial representations of $GL(V)$

Fix an algebraically closed char 0 field $\mathbb{F}$ and let V be an n-dimensional vector space over $\mathbb{F}$. A choice of a basis in V gives an identification of $GL(V)$ w. $GL_n(\mathbb{F})$ and defines the n^2 distinguished functions of $GL(V)$, the matrix entries x_{ij}, $i,j=1,\dots n$. Similarly, for a representation of $GL(V)$ in a space W (say, of $\dim = m$), we can choose a basis in W identifying $GL(W)$ w. $GL_m(\mathbb{F})$. The representation gives us m^2 functions on $GL(V)$, its matrix coefficients.

Definition: Let $d \in \mathbb{Z}_{\geq 0}$. We say W is a polynomial representation of degree d if its matrix coefficients are homogeneous degree d polynomials in the x_{ij}'s.

Exercise: • Show that this is well-defined (independent of the choices of bases in V & W).

• Direct sums, sub- and quotient representations of a polynomial degree d representation are polynomial of degree d.

Example: V itself is a polynomial representation of degree 1. $V^{\otimes d}$ is polynomial of degree d (more generally, if W_1, W_2

are polynomial representations of degrees d_1, d_2, then $W_1 \otimes W_2$ is a polynomial representation of degree d_1+d_2).

1.2) Schur-Weyl duality

The starting observation here is that $V^{\otimes d}$ also carries a representation of S_d: $\sigma(v_1 \otimes \ldots \otimes v_d) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(d)}$. It commutes with the representation of $GL(V)$ ($g(v_1 \otimes \ldots \otimes v_d) = gv_1 \otimes gv_2 \otimes \ldots \otimes gv_d$) giving us a representation of $GL(V) \times S_d$.

This gives a way of constructing more polynomial representations of $GL(V)$. Namely, let U be a representation of S_d. The representation of $GL(V)$ in $V^{\otimes d}$ gives rise to a representation in $\mathrm{Hom}_{S_d}(U, V^{\otimes d})$ (by acting on the target), it's also polynomial of deg d (exercise).

For a partition λ of d define

$$S^\lambda(V) := \mathrm{Hom}_{S_d}(V_\lambda, V^{\otimes d}).$$

Recall from Lecture B1 that $V_\lambda = (\mathbb{F}S_d)\varepsilon_\lambda$, where ε_λ is the Young symmetrized. By Lemma in Sec 1 of that lecture $S^\lambda(V) = \varepsilon_\lambda(V^{\otimes d})$.

Examples: • Let $\lambda=(d)$. Then ε_λ is the averaging idempotent $\varepsilon_\lambda = \frac{1}{|S_n|}\sum_{\sigma\in S_n}\sigma$ and $S^\lambda(V)$ consists of symmetric tensors, i.e $S^\lambda(V)$ is the dth symmetric power, $S^d(V)$.

• Similarly, for $\lambda=(1,1,\dots,1)$, $\varepsilon_\lambda = \frac{1}{|S_n|}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\sigma$, and $S^\lambda(V)$ consists of skew-symmetric tensors, i.e. $S^\lambda(V)$ is the dth exterior power.

In general, the structure of $S^\lambda(V)$ is not easy to describe.

Here's the main result of Schur-Weyl duality, see Secs 5.18-5.19 in [E].

Theorem: 1) The representation $S^\lambda(V)$ of $GL(V)$ is irreducible if $\lambda_1^t \leq n$ or 0 else.

2) We have $V^{\otimes d} \simeq \bigoplus_\lambda S^\lambda(V)\otimes V_\lambda$, the decomposition into the direct sum of irreducible $GL(V)\times S_d$-modules.

In fact, 1) is not fully proved in [E]. Let's show that $S^\lambda(V)=\{0\}$ if $\lambda_1^t > n$.

Lemma: For representations U_i of S_{d_i}, $i=1,2$, we have an isomorphism of $GL(V)$-representations:

$$\mathrm{Hom}_{S_{d_1+d_2}}\left(\mathrm{Ind}_{S_{d_1}\times S_{d_2}}^{S_{d_1+d_2}} U_1\otimes U_2,\ V^{\otimes d_1+d_2}\right)$$
$$\simeq \mathrm{Hom}_{S_{d_1}}(U_1, V^{\otimes d_1})\otimes \mathrm{Hom}_{S_{d_2}}(U, V^{\otimes d_2})$$

Proof: We use the Frobenius reciprocity in the form, where Ind is left adjoint to Res (see Bonus to Lec 14). Apply this to the l.h.s. of the desired isomorphism to get:

$$\mathrm{Hom}_{S_{d_1}\times S_{d_2}}(U_1\otimes U_2,\ V^{\otimes(d_1+d_2)}).$$

To identify this with the r.h.s. is an exercise $\square$

In particular, $\mathrm{Hom}_{S_d}(I_\lambda^-, V^{\otimes d}) = \bigoplus_{j=1}^{\lambda_1}(\Lambda^{\lambda_j^t} V)$. If $\lambda_1^t > n$, then $\Lambda^{\lambda_j^t} V = \{0\}$, hence $\mathrm{Hom}_{S_d}(I_\lambda^-, V^{\otimes d}) = \{0\}$ and so $S^\lambda(V) = \mathrm{Hom}_{S_d}(V_\lambda, V^{\otimes d}) = \{0\}$.

In fact, if $\lambda_1^t \leq n$, then $S^\lambda(V) \neq \{0\}$ but this is harder.