Lecture B3: More on representations of symmetric groups, pt. 3
Schur polynomials.

1) Definition and basic properties.
2) Connections to Representation theory

1) Definition and basic properties.

Lecture 18 hints at the connection between representations of symmetric groups and symmetric polynomials. The goal of this bonus lecture is to elaborate on this connection. A special role in this connection - and other connections to Representation theory - is played by a remarkable family of symmetric polynomials known as the Schur polynomials. A basic reference here for us is:

[F]: W. Fulton "Young tableaux. With applications to Representation theory & Geometry."

1.1) Equivalent definitions.

The first definition uses Kostka numbers mentioned in the end of Sec 1.2 in Lec 18. Let $K_{\lambda\mu}$ be the Kostka number, i.e. the number of Young tableaux of shape λ and weight μ. Fix $n \in \mathbb{Z}_{\geqslant 0}$ and choose $N \geqslant n$. Define the monomial symmetric polynomial m_λ as the sum of all monomials of the form $x_{\tau(1)}^{\lambda_1} x_{\tau(2)}^{\lambda_2} \dots x_{\tau(N)}^{\lambda_N}$ for $\tau \in S_N$ (w. multiplicity 1). For example, for $\lambda = (n)$, we have $m_\lambda = p_n$ (the power symmetric polynomial), while for $\lambda = (1, \dots, 1)$, we have $m_\lambda = e_n$ (the elementary symmetric polynomial).

Definition: Let λ be a partition of n. Define the Schur polynomial s_λ as $\sum_\mu K_{\lambda\mu} m_\mu$, where the sum is taken over all partitions μ of n.

Example: Let $\lambda = (n)$. Then $K_{\lambda\mu} = 1$ for all μ. So $s_{(n)}$ is the "complete symmetric polynomial": the sum of all deg n monomials w. coefficient 1, e.g

$$s_{(2)}(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_1x_3.$$

Now let $\lambda=(1,\dots 1)$ (n times). Then $s_\lambda(x_1,\dots x_N)=e_n(x_1,\dots x_N)$, exercise.

Now note that m_μ's form a basis in the abelian group $\mathbb{Z}[x_1,\dots,x_N]_n^{S_N}$ of deg n homogeneous symmetric polynomials w. integral coefficients. Note also that $K_{\lambda\lambda}=1$ & $K_{\lambda\mu}\neq 0 \Rightarrow \lambda \geq \mu$ in the dominance order (i.e $\sum_{i=1}^{k}\lambda_i \geq \sum_{i=1}^{k}\mu_i\ \forall k$). It follows that the polynomials s_λ also form a basis in $\mathbb{Z}[x_1,\dots,x_N]_n^{S_N}$.

Now we are going to give a definition of a very different nature. We can talk about anti-symmetric polynomials in $\mathbb{Z}[x_1,\dots x_N]$: those that switch the sign when we permute x_i & x_j for $i\neq j$. Every such polynomial is divisible by x_i-x_j and, since $\mathbb{Z}[x_1,\dots x_N]$ is a UFD by $\prod_{i<j}(x_i-x_j)=\Delta$ (the Vandermonde).

Here's a relevant example of an anti-symmetric polynomial: $\det(x_i^{\lambda_j+n-j})_{i,j=1}^{N}$, where $\lambda=(\lambda_1,\dots,\lambda_N)$ is a partition of n. So $\det(x_i^{\lambda_j+n-j})/\Delta$ is a polynomial, in fact, symmetric.

The following result is known as the Jacobi-Trudi formula:

Theorem: $s_\lambda(x_1,\ldots,x_N)=\det(x_i^{\lambda_j+n-j})/\Delta$

For a sketch of proof and references see Sec 6.1 in [F].

1.2) Bilinear form

Since the s_λ's form a basis in $\mathbb{Z}[x_1,\ldots,x_N]_n^{S_N}$ we have the unique $\mathbb{Z}$-bilinear symmetric form, $(\cdot,\cdot)$, on this abelian group s.t. $(s_\lambda,s_\mu)=\delta_{\lambda\mu}$.

We want to compute this pairing on some other symmetric polynomials. In Lec 17 we have introduced the polynomials $p_\lambda=p_{\lambda_1}p_{\lambda_2}\ldots p_{\lambda_k}$, where $\lambda=(\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_k>0)$ is a partition of n. Another family we will need is h_λ. Here

$$h_\lambda=s_{(\lambda_1)}\ldots s_{(\lambda_k)}.$$

The following is Proposition 3 in Sec 6.2 of [F].

Theorem: We have $(p_\lambda,p_\mu)=z(\lambda)\delta_{\lambda\mu}$, where $z(\lambda)$ is the order of the centralizer of an element w. cycle type λ in S_n (see Exercise in Sec 1.3 of Lec 18). Also $(h_\lambda,m_\mu)=\delta_{\lambda\mu}$.

1.3) Remark on the number of variables.

Above we considered the situation, where the number of variables, N, is $\geqslant n$, the degree. We can omit this restriction and still consider the polynomials, $m_\lambda, s_\lambda, h_\lambda$ etc, with the same definitions. Note that $m_\lambda(x_1,\dots x_N)=0$ if λ has more than N parts, while the polynomials $m_\lambda(x_1,\dots x_N)$, where λ has $\leqslant N$ parts, form a basis in $\mathbb{Z}[x_1,\dots x_N]^{S_N}$.

Exercise: The polynomials $s_\lambda(x_1,\dots x_N)$, where λ has $\leqslant N$ parts, form a basis in $\mathbb{Z}[x_1,\dots x_N]^{S_N}$ (in particular, are nonzero).

2) Connections to Representation theory

2.1) Symmetric groups.

Let $N\geqslant n$. We define the Frobenius character F_V of a representation V of S_n to be the following deg n element of $\mathbb{Z}[x_1,\dots x_N]^{S_N}$: if n_λ is the multiplicity of V_λ in V, then

$$(*)\quad F_V=\sum_\lambda n_\lambda s_\lambda(x_1,\dots x_N),$$

where the sum is taken over all partitions of λ of n.

One can ask why to give this definition. Here's some way to answer.

Proposition: Let V_1, V_2 be representations of S_{n_1}, S_{n_2} w. $n+n=n$, we can $V_1 \otimes V_2$ as a representation of $S_{n_1}\times S_{n_2}$. Set $V = \mathrm{Ind}_{S_{n_1}\times S_{n_2}}^{S_n} V_1 \otimes V_2$ Then

(1) $$F_V = F_{V_1} F_{V_2}.$$

Proof: We start by showing that $F_{I^+_\lambda} = h_\lambda$. Recall, Remark in Sec 1.2 of Lec 18, that

$$I^+_\lambda \simeq \bigoplus_\mu V_\mu^{\oplus K_{\mu\lambda}}.$$

So, by (*), $F_{I^+_\lambda} = \bigoplus_\mu K_{\mu\lambda} S_\mu$. Recall that $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$, and, by Sec 1.2, $(h_\lambda, m_\mu) = (s_\lambda, s_\mu) = \delta_{\lambda\mu}$. This implies that the coefficient of m_μ in s_λ is equal to the coefficient of s_λ in h_μ for all λ, μ. So $F_{I^+_\lambda} = h_\lambda$ for all λ.

This equality implies (1) when $V_1 = I^+_{\lambda^1}$, $V_2 = I^+_{\lambda^2}$ for arbitrary partitions λ^1 of n_1 & λ^2 of n_2. Notice that we can express the character of arbitrary V_1 as a $\mathbb{Z}$-linear combination of characters of $I^+_{\lambda^1}$'s (it's enough to show this for

$V_1 = V_{\mu'}$ for some μ', here it follows from the observation that the matrix $(K_{\lambda'\mu'})$ is uni-triangular for a suitable order on partitions). The same, of course, works for V_2. This observation allows to deduce (1) for arbitrary V_1, V_2 to the case $V_1 = I^+_{\lambda^1}$, $V_2 = I^+_{\lambda^2}$. To make this rigorous in an exercise, you may note that the assigment $V \mapsto F_V$ factors through $K_0(Rep(S_n))$ and the classes $[I^+_\lambda]$ form a basis in the free abelian group $K_0(Rep(S_n))$ (for details on K_0 see the bonus part of Lec 11). $\square$

2.2) General linear groups.

Here we are in the setting of Lecture B2. Assume, for simplicity, that the base field in $\mathbb{C}$.

Recall from Lec 22, that to a partition λ of d we can assign the polynomial degree d representation $S^\lambda(V)$ of $GL(V)$.

Assume $\dim V = m$. We claim that we can interpret the character χ_U of a polynomial deg d representation U as an element of $\mathbb{Z}[x_1, \ldots x_m]^{S_m}$. Observe that the diagonalizable matrices are dense in $GL_m(\mathbb{C})$. Since the matrix coefficients of U are polynomials in the matrix entries, so is the character. Therefore, it's uniquely determined by its restriction to any dense subset of $GL_m(\mathbb{C})$, in particular, to the diagonalizable matrices.

Now we use that χ_U is conjugation invariant. So it's uniquely determined by its restriction to diagonal matrices. This restriction is a polynomial in the diagonal matrix entries, and it's symmetric: diagonal matrices that differ by a permutation of entries are conjugate. This is how we view χ_U as a symmetric polynomial.

Examples: Let T denote the subgroup of diagonal matrices, and $v_1, \dots v_m$ be the tautological basis in V so that for $t = \mathrm{diag}(t_1, \dots t_m) \in T$ we have $t.v_i = t_i v_i$.

1) $\chi_V \longleftrightarrow \sum_{i=1}^{m} x_i$

2) Since $\chi_{V^{\otimes d}} = \chi_V^d$, $\chi_{V^{\otimes d}} \longleftrightarrow \left(\sum_{i=1}^{m} x_i\right)^d$

3) Let $U = S^d(V)$. It has basis $v_1^{d_1} \dots v_m^{d_m}$ $(d_1 + \dots + d_m = d)$ & $t.(v_1^{d_1} \dots v_m^{d_m}) = t_1^{d_1} \dots t_m^{d_m}$. It follows that $\chi_{S^d(V)}$ corresponds to

$\sum_{(d_1,\dots d_m)} x_1^{d_1}\dots x_m^{d_m}$, the complete symmetric polynomial, i.e. $s_{(d)}(x_1,\dots x_m)$.
Recall (Sec 1.2 of Lec B2) that $S^{(d)}(V)=S^d(V)$.

4) Let $U=\Lambda^d(U)$. It has basis $v_{i_1}\wedge v_{i_2}\wedge\dots\wedge v_{i_d}$ w. $i_1<i_2<\dots<i_d$.
So $\chi_{\Lambda^d(V)}\longleftrightarrow e_d(x_1,\dots x_m)$. Recall that, for $\lambda=(1,1,\dots,1)$, we have
$s_\lambda=e_d$ & $S^\lambda(V)=\Lambda^d(V)$.

The following result, Theorem 5.23.2, relates $\chi_{S^\lambda(V)}$ to the Schur polynomials generalizing the example.

Theorem: $s_\lambda(x_1,\dots x_m)$ is the symmetric polynomial corresponding to $S^\lambda(V)$.

Exercise: Use Exercise in Sec 1.3 to conclude that $S^\lambda(V)\neq\{0\}$ if λ has $\leq m$ parts.