MATH 720, Lecture 1.

1) Unitary representations
2) Motivation: symmetries in Quantum Mechanics.
3) Orbit method.
4) Poisson manifolds.

The topics of this course is singular symplectic varieties, their quantizations & applications to Lie theory: understanding "Harish-Chandra bimodules" and more general "Harish-Chandra modules." This circle of ideas has a bunch of motivations (from Rep. theory, Alg. geometry & Math. Physics). Today we will briefly discuss the oldest of them: unitary representations of groups & symmetry in Quantum Mechanics. Note that this discussion is for motivational purposes only.

1) Unitary representations.

The first setting of Representation theory one usually encounters is finite group representations. Here's one of the basic general results.

Thm: Let G be a finite group & V be a finite dimensional G-representations (over $\mathbb{C}$). Then V admits a G-invariant Hermitian scalar product, say $(\cdot,\cdot)$.

Note that G acts on V by unitary (w.r.t. $(\cdot,\cdot)$) operators. So we say V is a unitary representation.

The first application of this theorem is the complete reducibility of V: if $U \subset V$ is a subrepresentation, then $U^\perp$ is a complimentary subrepresentation.

A natural question is to which extent the theorem extends to infinite groups. Of special interest are Lie groups (where we consider C^∞-representations).

A classical result is that the complete analog of the theorem holds when G is a compact Lie group (as we can still average over G). On the other hand, the only unitary irreducible representation of a non-compact simple Lie group is the trivial one.

An interesting and important question is to understand infinite dimensional unitary representations. Let's recall/give relevant definitions.

Def'n: 1) A Hilbert space is a $\mathbb{C}$-vector space equipped with a Hermitian scalar product s.t. the resulting topology is complete.

2) A unitary representation of a Lie group G is a representation of G in a Hilbert space V by unitary operators s.t. the action map $G \times V \to V$ is continuous.

3) A subrepresentation in V is a <u>closed</u> G-stable subspace. This allows us to talk about <u>irreducible</u> unitary representations.

Note that every unitary representation is completely reducible and splits as an infinite "direct sum" (more precisely, its topological version) of irreps.

Examples: 1) Here's a classical example of a unitary rep'n. Let M be a manifold with fixed volume form. Then it makes sense to speak about $L^2(M)$, the space of squareintegrable functions. It's a Hilbert space.

If G acts on M preserving the volume form, then it also acts on $L^2(M)$ turning it into a unitary representation.

An important special case: $M = G$ w. left-invariant volume form (defined uniquely up to proportionality) and action of G by left translations).

2) If G is compact, then every unitary irrep is finite dimensional. This yields the classification of unitary irreps (for compact Lie groups).

Here's one of the motivating questions for what we are going to study in this course (but which we aren't going to attempt to answer):

Problem: Given a semisimple Lie group G classify its unitary irreps.

2) Motivation: symmetry in Quantum Mechanics.

Here's some Quantum Mechanics terminology (whose actual meaning we don't need):

- Space of states: a Hilbert space, V.
- Observables: Hermitian operators on V.
- Hamiltonian: an observable H.
- An evolution equation that explains how a given observable changes on the trajectory of the system - Heisenberg equation $\dot{F}(t)=\frac{i}{\hbar}[H,F(t)]$, where $\hbar$ is the Plank constant
- Symmetry: a unitary representation of a reasonable group (e.g. a Lie group) preserving H.

So unitary representations arise as symmetry of quantum mechanical systems. Their usefulness is that systems with more symmetries are easier to "integrate." Classifying the unitary irreps allows to understand quantum mechanical systems w. symmetries.

3) Orbit method

There's a class of Lie groups for which there's a very conceptually (but not computationally) satisfactory answer to the classification question in Sec 1: nilpotent groups (an example: the group of uni-triangular matrices, $\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix}$). The result can be extended to a somewhat wider class of groups but we'll restrict ourselves to nilpotent ones. The answer, due to Kirillov, relates unitary irreps to "coadjoint orbits" and is known as the Orbit method.

Let's explain what the coadjoint orbits are. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then G acts on $\mathfrak{g}$ via the adjoint representation and hence on $\mathfrak{g}^*$ via the coadjoint representation.

Def'n: By a coadjoint orbit we mean an orbit in the coadjoint representation $\mathfrak{g}^*$.

Example: Let $G = SL_n(\mathbb{C})$ (viewed as a real Lie group). Then $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and the adjoint representation is by matrix conjugations: $\mathrm{Ad}(g)x = gxg^{-1}$. Note that $\mathfrak{g}$ comes with the trace form: $(x,y) := \mathrm{tr}(xy)$, which is G-invariant & orthogonal. It allows to identify $\mathfrak{g}$ & $\mathfrak{g}^*$ as G-representations. So the coadjoint orbits for G is the same thing as the conjugacy classes of traceless matrices. Their classification is given by the Jordan normal form thm.

Thm (Kirillov '61) Suppose G is nilpotent & simply connected. Then $\exists$ natural bijection between the following two sets:

- Unitary G-reps (up to iso).
- The coadjoint G-orbits.

We won't need the construction. What's important for us is that for semisimple groups, there's no natural bijection. For example, if G is compact, then the unitary irreps (= finite dimensional irreps) are classified by dominant <u>integral</u> weights. The coadjoint orbits are classified by dominant <u>real</u> weights.

Still one expects that there is a connection between unitary irreps & coadjoint orbits morally grounded in the "correspondence principle" between Quantum and Classical Mechanics to be discussed in the next lecture.

4) Poisson manifolds.

Let $\mathbb{F}$ be a field and A be a commutative associative $\mathbb{F}$-algebra with 1. By a bracket on A we mean a skew-symmetric bilinear map $\{\cdot,\cdot\}: A\times A \to A$ satisfying the Leibniz identity:

skew-symmetry

$$\{a, bc\} = \{a,b\}c + \{a,c\}b \iff \{bc,a\} = \{b,a\}c + \{c,a\}b.$$

Definition: $\{\cdot,\cdot\}$ is a Poisson bracket if it satisfies the Jacobi identity (and so A becomes a Lie algebra).

In particular, we can apply this definition to the situation when $\mathbb{F} = \mathbb{R}$ and $A = C^\infty(M)$ for a C^∞-manifold M. To give a bracket on $C^\infty(M)$ is equivalent to giving a bivector field, a C^∞-section, $\mathcal{P}$, of $\Lambda^2 T_{\mathcal{M}}$: $\{f,g\}_{\mathcal{P}} := \langle \mathcal{P}, df \wedge dg \rangle$. If $\{\cdot,\cdot\}_{\mathcal{P}}$ is Poisson, then we say that $\mathcal{P}$ is a Poisson bivector and $(\mathcal{M}, \mathcal{P})$ (or just $\mathcal{M}$) is a Poisson manifold.

Definition: We say $\mathcal{P}$ is nondegenerate if $\mathcal{P}_m \in \Lambda^2 T_{\mathcal{M},m}$ is nondegenerate $\forall\, m \in \mathcal{M}$. Then we identify $T^*_{\mathcal{M},m} \xrightarrow{\sim} T_{\mathcal{M},m}$ via $\langle \mathcal{P}_m, \cdot \rangle$ and get $\omega \in \Gamma(\Lambda^2 T^*_{\mathcal{M}})$ out of $\mathcal{P}$.

The condition that $\mathcal{P}$ is Poisson is equivalent to ω being symplectic i.e. nondegenerate (automatic from the construction) & closed.