Lecture 10.

1) Where we are.

1) Nilpotent cone.

2) Categorical quotient for $\mathfrak{g}_c \otimes \mathfrak{g}$ & Poisson deformations

Ref: [B], Ch. 8, Secs 8,10; [CG], Sec. 3.2; [E], Ch. 17 & 18; [Ko]

0) Where we are. We have defined positively graded Poisson algebras $A (A_0 = \{0\}, A_0 = \mathbb{C})$ and posed the problems of classifying their filtered Poisson deformations & filtered quantizations. We are only going to approach this problem in the case when $A = \mathbb{C}[x]$ for a conical symplectic singularity $X$. Our main example of $X$ is $\text{Spec } \mathbb{C}[\tilde{O}]$, where $\tilde{O}$ is a $\mathbb{C}$-equivariant cover of a nilpotent orbit $O_{cg}$.

In this lecture we will concentrate on the special case, where $\tilde{O} = O$ is the so called principal nilpotent orbit, and $X$ is the "nilpotent cone." We will produce examples of filtered Poisson deformations; filtered quantizations are for the next lecture.
1) Nilpotent cone.

Let \( G \) be a semisimple algebraic group & \( g = \text{Lie}(G) \).

Definition: The nilpotent cone \( N := \{ x \in g | x \text{ is nilpotent} \} \).

Example: For \( g = \mathfrak{sl}_n \), we get \( N = \{ x \in g | x_k(x) = 0, k=2,\ldots,n \} \), where \( x_k(x) = \text{coeff of } t^k \text{ in the char. polynomial } \det(x-tI) \).

Exercise 1: \( N \) is a closed subvariety for any \( g \).

Theorem: \( \exists! \) nilpotent orbit \( O \subset g \text{ s.t. } N = \overline{O} \). It's called principal and has \( \dim = \dim g - \text{rk } g \) (\( \text{rk } g = \dim \mathfrak{g} \)).

Example: \( g = \mathfrak{sl}_n \): principal \( \Rightarrow \) single Jordan block.

Proof: Step 1: Let \( g = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{p} \) be the triangular decomposition. We claim that \( N = \mathfrak{g}_\mathfrak{n} \). Indeed, let \( e \in N \). The subalgebra \( \mathfrak{g}_e \subset \mathfrak{g} \) is abelian, hence solvable, hence conjugate to a subalgebra in the Borel \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \). But \( \mathfrak{n} = \{ x \in \mathfrak{b} | x \text{ is nilpotent} \} \), so \( e \) is conjugate to an element of \( \mathfrak{n} \) \( \Leftrightarrow \) \( N = \mathfrak{g}_\mathfrak{n} \).
Step 2: Define an $\mathfrak{sl}_2$-triple $(e, h, f)$ in $g$ as follows. Let
\[ \Pi \subset \Delta_+ \subset \mathfrak{h}^* \] be systems of simple & positive roots. Set \[ h = \sum_{\alpha \in \Pi} \alpha. \] Define $r_\beta (\beta \in \Pi)$ as the coefficient of $\beta^\vee$ in $h$ so that $h = \sum_{\beta \in \Pi} r_\beta \beta$. Choose $\mathfrak{sl}_2$-triples $(e_\beta, h_\beta, f_\beta)$, $\beta \in \Pi$ with $e_\beta \in \mathfrak{g}_\beta$, $f_\beta \in \mathfrak{g}_{-\beta}$ (root subspaces) and set
\[ e = \sum_{\beta \in \Pi} \sqrt{r_\beta} e_\beta, \quad f = \sum_{\beta \in \Pi} \sqrt{r_\beta} f_\beta. \]

Exercise 2: [e.g. [8], Ch. 8, Sec. 10.4] 1) $\langle \beta, h \rangle = 2 \quad \forall \beta \in \Pi$
2) $(e, h, f)$ is indeed an $\mathfrak{sl}_2$-triple.

Set $\mathfrak{o}_\beta := \mathfrak{c}_e$.

Step 3: Consider the grading $g = \bigoplus \mathfrak{g}_i$ by eigenvalues of $h$.

Exercise 3: $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_{\geq 1} = \mathfrak{n}$.

Consider the variety $Y = \mathbb{C} \times \mathfrak{p}_{\mathfrak{g}_\mathfrak{p}} = \{ P = B, \text{ the Borel} \} = \mathbb{C} \times \mathfrak{n}$ from Sec 3 of Lec 8. By the theorem in that section, $Y \xrightarrow{\pi} \overline{\mathfrak{o}_\beta}$ is a resolution of singularities $\Rightarrow \dim \overline{\mathfrak{o}_\beta} = \dim Y$
\[ = \dim g - \dim \mathfrak{b} + \dim \mathfrak{n} = \dim g - \operatorname{rk} g. \] Also, $\operatorname{im} \pi = \mathbb{C} \mathfrak{n}$ and by Step 1, $\overline{\mathfrak{o}_\beta} = \operatorname{im} \pi = \mathfrak{n}$. \[ \square \]
Remarks: 1) Kostant proved that \( N \) is normal, we’ll comment on this later. So \( C[N] = C[\overline{O}_{pr}] = C[O_{pr}] \)  
(normalization of the 2nd term.)

2) Note that \( n = g^\perp \) w.r.t. Killing form. So \( G \times g^n \cong G \times g^*(g/b)^* = T^*(g/b) \). This is a symplectic variety & \( G \times g^*(g/b) \) is Hamiltonian (Sec 2 of Lec 2) and \( \pi: T^*(g/b) \to g^*(\cong g^*) \) is the moment map (exercise). The map \( \pi: T^*(g/b) \to N \) is an example of a symplectic resolution. It’s known as the Springer resolution – it’s one of the most important morphisms in the geometric representation theory.

2) Categorical quotient for \( G/N \) & Poisson deformations

Consider the quotient morphism \( \pi_G: g \to g//G \). We’ll see that:

- For \( a \in g//G \), \( C[\pi^{-1}_G(a)] \) carries a natural Poisson algebra structure.
- \( C[\pi_0^{-1}(0)] \cong C[O_{pr}] \), a graded Poisson algebra isomorphism
- \( C[\pi^{-1}_G(a)] \) can be viewed as a filtered Poisson deformation of \( C[\pi_0^{-1}(0)] \).

We start w. following classical result.
Theorem (Chevalley, [B], Ch. 8, Sec 8): Let \( \mathfrak{g} \subseteq g, W \subseteq G(\mathfrak{g}) \) be Cartan subalgebra & Weyl group. Then the restriction homomorphism \( C[\mathfrak{g}] \rightarrow C[g] \) restricts to \( C[\mathfrak{g}]^W \rightarrow C[g]^W \).

Recall (Chevalley- Shephard-Todd Thm from Sec 2.2 of Lect 9) that \( g/W \) is smooth. \( C[\mathfrak{g}]^W \) is positively graded, so \( C[\mathfrak{g}]^W \) is the algebra of polynomials in \( n \) \( g \) homogeneous elements, say \( f_1, \ldots, f_r \).

2.1) Deformations of \( C[N] \).

Proposition: The following are true:

(i) \( N = \pi_\mathfrak{g}^{-1}(0) \) as subsets of \( g \).

(ii) \( f_1, \ldots, f_r \in C[\mathfrak{g}] \) form a regular sequence (Sec 1.1 in Lect 9)

(iii) \( C[\mathfrak{g}] \) is a free graded \( C[\mathfrak{g}]^F \)-module.

(iv) \( \pi_\mathfrak{g}^{-1}(0) \) is reduced and normal as a scheme.

Sketch of proof:

(i): \( N \subseteq \pi_\mathfrak{g}^{-1}(0) \); in the proof in Sec 2.2 of Lect 7 (after Exer 7) we've see that \( te \in C \Rightarrow \pi_\mathfrak{g}(te) = \pi_\mathfrak{g}(e) \neq 0 \) \( t \in C^* \Rightarrow \pi_\mathfrak{g}(e) = 0 \).
Let \( p: \mathfrak{g} \rightarrow \mathfrak{sl}(V) \) be a faithful representation.

Let \( X_{\mathfrak{g}} \), \( -X_{\mathfrak{g}} \) be the coeff. of the char. polynomial of \( p(x), x \in \mathfrak{g} \).

Then \( X_{\mathfrak{g}} \in \text{max ideal of } 0 \text{ in } \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}. \) So \( X_{\mathfrak{g}}(x) = 0 \) for \( x \in \pi_\mathfrak{g}^{-1}(0) \).

Hence \( x \) acts on \( V \) by a nilpotent operator \( x \in N \).

(ii) Follows from (i) & \( \text{codim}_{\mathfrak{g}} \pi_\mathfrak{g}^{-1}(0) = \text{codim}_{\mathfrak{g}} N = k \).

(iii) We'll use the following important fact (vanishing of the 1st Koszul homology):

**Fact [CE, Cor. 17.5]:** Let \( R \) be a Noetherian commutative ring. Suppose \( l, \ldots, k \in R \) is a regular sequence, and \( b_1, \ldots, b_k \in \mathfrak{R} \) are s.t. \( \sum_{i=1}^{k} f_i b_i = 0. \) Then \( \exists \ b_{ij} \in R \) w. \( b_{ii} = 0, b_{ij} = -b_{ji} \) s.t. \( b_i = \sum_{j=1}^{k} b_{ij} f_j. \)

Now we get back to (iii). The algebra \( \mathbb{C}[\pi_\mathfrak{g}^{-1}(0)] \) is graded.

Pick a homogeneous vector space basis \( b_i \in \mathbb{C}[\pi_\mathfrak{g}^{-1}(0)], i \in I, \)

& lift it to homogeneous \( b_i \in \mathbb{C}[\mathfrak{g}] \). By the graded Nakayama lemma, \( b_i (i \in I) \) span the \( \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \)-module \( \mathbb{C}[\mathfrak{g}] \), while Fact
implies they are linearly independent (exercise).

iv) By (ii), $\pi_{\xi}^{-1}(0)$ is a complete intersection. By Serre's normality criterion ([E, Thm 18.5]) we need to show that 
\[ \{ x \in N | d_x \pi_{\xi} \text{ is not surjective} \} \text{ has codim} = 2 \text{ in } N. \] 
Kostant proved, [Ko], that $d_x \pi_{\xi}$ is surjective if $x \in O_{pr}$. Now we use that $\text{codim}_N N \backslash O_{pr} = 2$ and finish the proof. \( \Box \)

Corollary: For $a \in g/W$ consider the filtration on $C[\pi_{\xi}^{-1}(a)]$ induced by the grading on $C[g]$. Then we have a natural isomorphism $C[\pi_{\xi}^{-1}(0)] \rightarrow \text{gr} C[\pi_{\xi}^{-1}(a)]$.

Proof: Set $a_i = f_i(a), i = 1 \ldots r$. Then $C[\pi_{\xi}^{-1}(a)] = C[g]/(f_i - a_i)^r$, $C[\pi_{\xi}^{-1}(0)] = C[g]/(f_i)$. We have the natural graded epimorphism $C[g] \twoheadrightarrow \text{gr} C[\pi_{\xi}^{-1}(a)]$ that sends the $f_i$'s to 0 so factors through $C[\pi_{\xi}^{-1}(0)] \twoheadrightarrow \text{gr} C[\pi_{\xi}^{-1}(a)]$. It sends the $b_i$'s from the proof of (iii) in Proposition to the image of the $b_i$'s in $C[\pi_{\xi}^{-1}(a)]$. Since $b_i$'s form a basis in the $C[g]$-module $C[g]$, their images in $C[\pi_{\xi}^{-1}(a)]$ form a basis, so (**) in an iso. \( \Box \)
2.2) Poisson structures.

Recall that:

(I) \( S(g) = C[g^*] \) has the unique Poisson structure with \( \{ \xi, \eta \} = \{ [\xi, \eta] \} \neq \xi, \eta \in g. \)

(II) If \( X \) is a Poisson variety and \( G \triangleleft X \) is a Hamiltonian action, then the comoment map \( \xi \mapsto H_\xi \) satisfies \( H_{[\xi, \eta]} = [H_\xi, H_\eta] \). So it extends to a Poisson algebra homomorphism \( C[g^*] \to C[X] \).

**Definition:** The center of the Poisson algebra \( A \) is \( \{ z \in A | [z, a] = 0 \ \forall \, a \in A \} \). It's a subalgebra.

**Exercise 1:** The center of \( C[g^*] \) is \( C[g^*]^G \) (hint: the former is \( \{ z \in C[g^*] | \{ z, \xi \} = 0 \ \forall \, \xi \in g \} \).

Now apply (II) to \( X = Q_{pr} \). The moment map \( \mu: Q_{pr} \to g^* \) factors as \( Q_{pr} \to \overline{Q}_{pr} = N \to g^* \). Since \( C[Q_{pr}] = C[\overline{Q}_{pr}] \), Rem 1 in Sec 1, we see that the epimorphism

\[ C[g^*] \to C[g^{*^{-1}}(0)] = C[Q_{pr}] \]

is Poisson.
Exercise 2: Let \( A \) be a Poisson algebra, \( Z \) be its center, \& \( I \subset Z \) an ideal. Then \( A/I \) carries the unique Poisson bracket st. \( A \rightarrow A/I \) is a Poisson algebra homomorphism.

Applying this to \( A = C[\mathfrak{g}^*] \) \& the maximal ideal \( I < C[\mathfrak{g}^*]^G \) of \( \mathfrak{g}^* / \mathfrak{g} \), we get a Poisson bracket on \( C[\mathfrak{g}^*]^G / C[\mathfrak{g}^*]^G I = C[\mathfrak{g}^{-1}(a)] \)

Exercise 3: 1) This Poisson bracket on \( C[\mathfrak{g}^{-1}(a)] \) has degree \( \leq -1 \) w.r.t. the filtration on \( C[\mathfrak{g}^{-1}(a)] \).
2) The isomorphism \( C[\mathfrak{g}^{-1}(a)] \rightarrow gr \ C[\mathfrak{g}^{-1}(a)] \) is Poisson.

So we have constructed a family of filtered Poisson deformations of \( C[N] \) parameterized by points of \( \mathfrak{g} / \mathfrak{g}_0 \sim \mathfrak{k} / \mathfrak{w} \).

The following exercise examines the structure on \( \mathfrak{g}^{-1}(a) \).

Exercise 4: 1) The unique closed \( G \)-orbit in \( \mathfrak{g}^{-1}(a) \) is \( s \)-simple.
2) \( \mathfrak{g}^{-1}(a) \) contains the unique open orbit, say \( O_{pr,a} \).
3) \( C[\mathfrak{g}^{-1}(a)] \rightarrow C[O_{pr,a}] \).