Lecture 10.

0) Where we are.

1) Nilpotent cone.

2) Categorical quotient for $G \curvearrowright \mathfrak{g}$ & Poisson deformations

Ref: [B], Ch. 8, Secs 8, 10; [CG], Sec. 3.2; [E], Ch. 17 & 18 ; [Ko]

0) Where we are. We have defined positively graded Poisson algebras A ($A_{<0}=\{0\}$, $A_0=\mathbb{C}$, $\deg\{\cdot,\cdot\}=-d$) and posed the problems of classifying their filtered Poisson deformations & filtered quantizations. We are only going to approach this problem in the case when $A=\mathbb{C}[X]$ for a conical symplectic singularity X. Our main example of X is Spec $\mathbb{C}[\tilde{\mathbb{O}}]$, where $\tilde{\mathbb{O}}$ is a G-equivariant cover of a nilpotent orbit $\mathbb{O}\subset\mathfrak{g}$.

In this lecture we will concentrate on the special case, where $\tilde{\mathbb{O}}=\mathbb{O}$ is the so called principal nilpotent orbit, and X is the "nilpotent cone". We will produce examples of filtered Poisson deformations; filtered quantizations are for the next lecture.

1) Nilpotent cone.

Let G be a semisimple algebraic group & $\mathfrak{g} = \mathrm{Lie}(G)$.

Definition: The nilpotent cone $\mathcal{N} := \{x \in \mathfrak{g} \mid x \text{ is nilpotent}\}$.

Example: For $\mathfrak{g} = \mathfrak{sl}_n$, we get $\mathcal{N} = \{x \in \mathfrak{g} \mid \chi_k(x) = 0, k = 2, \dots n\}$, where $\chi_k(x) =$ coeff't of t^k in the char. polynomial $\det(x - tI)$.

Exercise 1: $\mathcal{N}$ is a closed subvariety for any $\mathfrak{g}$.

Theorem: $\exists!$ nilpotent orbit $\mathbb{O} \subset \mathfrak{g}$ s.t. $\mathcal{N} = \overline{\mathbb{O}}$. It's called principal and has $\dim = \dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}$ ($\mathrm{rk}\, \mathfrak{g} = \dim \mathfrak{h}$).

Example: $\mathfrak{g} = \mathfrak{sl}_n$: principal $\Leftrightarrow$ single Jordan block.

Proof: Step 1: Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the triangular decomposition. We claim that $\mathcal{N} = G\mathfrak{n}$. Indeed, let $e \in \mathcal{N}$. The subalgebra $\mathbb{C}e \subset \mathfrak{g}$ is abelian, hence solvable, hence conjugate to a subalgebra in the Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. But $\mathfrak{n} = \{x \in \mathfrak{b} \mid x \text{ is nilpotent}\}$, so e is conjugate to an element of $\mathfrak{n} \Leftrightarrow \mathcal{N} = G\mathfrak{n}$.

Step 2: Define an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{g}$ as follows. Let $\Pi \subset \Delta_+ \subset \mathfrak{h}^*$ be systems of simple & positive roots. Set $h = \sum_{\alpha\in\Delta_+} \alpha^\vee \in \mathfrak{h}$. Define r_β $(\beta\in\Pi)$ as the coefficient of $\beta^\vee$ in h so that $h = \sum_{\beta\in\Pi} r_\beta \beta^\vee$. Choose $\mathfrak{sl}_2$-triples $(e_\beta, \beta^\vee, f_\beta)$, $\beta\in\Pi$ w. $e_\beta \in \mathfrak{g}_\beta$, $f_\beta \in \mathfrak{g}_{-\beta}$ (root subspaces) and set

$$e = \sum_{\beta\in\Pi} \sqrt{r_\beta}\, e_\beta,\quad f = \sum_{\beta\in\Pi} \sqrt{r_\beta}\, f_\beta.$$

Exercise 2 (e.g. [B], Ch. 8, Sec. 10.4) 1) $\langle\beta, h\rangle = 2$ $\forall \beta\in\Pi$

2) (e,h,f) is indeed an $\mathfrak{sl}_2$-triple.

Set $\mathbb{O}_{pr} := Ge$.

Step 3: Consider the grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ by eigenvalues of h.

Exercise 3: $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_{\geq 1} = \mathfrak{g}_{\geq 2} = \mathfrak{n}$.

Consider the variety $Y = G\times^P \mathfrak{g}_{\geq 2} = [P = B$, the Borel$] = G\times^B \mathfrak{n}$ from Sec 3 of Lec 8. By the theorem in that section, $Y \xrightarrow{\pi} \overline{\mathbb{O}}_{pr}$ is a resolution of singularities $\Rightarrow \dim \overline{\mathbb{O}}_{pr} = \dim Y$ $= \dim\mathfrak{g} - \dim\mathfrak{b} + \dim\mathfrak{n} = \dim\mathfrak{g} - \operatorname{rk}\mathfrak{g}$. Also $\operatorname{im}\pi = G\mathfrak{n}$ and by Step 1, $\overline{\mathbb{O}}_{pr} = \operatorname{im}\pi = \mathcal{N}$. $\square$

Remarks: 1) Kostant proved that $\mathcal{N}$ is normal, we'll comment on this later. So $\mathbb{C}[\mathcal{N}] = \mathbb{C}[\overline{\mathbb{O}}_{pr}] = \mathbb{C}[\mathbb{O}_{pr}]$. normalization of the 2nd term.

2) Note that $\mathfrak{n} = \mathfrak{b}^\perp$ w.r.t. Killing form. So $G\times^B \mathfrak{n} \simeq G\times^B (\mathfrak{g}/\mathfrak{b})^* = T^*(G/B)$. This is a symplectic variety & $G \curvearrowright T^*(G/B)$ is Hamiltonian (Sec 2 of Lec 2) and $\pi: T^*(G/B) \to \mathfrak{g}\ (\simeq \mathfrak{g}^*)$ is the moment map (exercise). The map $\pi: T^*(G/B) \twoheadrightarrow \mathcal{N}$ is an example of a symplectic resolution. It's known as the Springer resolution - it's one of the most important morphisms in the geometric representation theory.

2) Categorical quotient for $G \curvearrowright \mathfrak{g}$ & Poisson deformations

Consider the quotient morphism $\pi_G: \mathfrak{g} \to \mathfrak{g}/\!\!/G$. We'll see that:

- $\forall\, a \in \mathfrak{g}/\!\!/G$, $\mathbb{C}[\pi_G^{-1}(a)]$ carries a natural Poisson algebra structure.
- $\mathbb{C}[\pi_G^{-1}(0)] \simeq \mathbb{C}[\mathbb{O}_{pr}]$, a graded Poisson algebra isomorphism.
- $\mathbb{C}[\pi_G^{-1}(a)]$ can be viewed as a filtered Poisson deformation of $\mathbb{C}[\pi_G^{-1}(0)]$.

We start w. following classical result.

Theorem (Chevalley, [B], Ch. 8, Sec 8): Let $\mathfrak{h} \subset \mathfrak{g}$, $W \subset GL(\mathfrak{h})$ be Cartan subalgebra & Weyl group. Then the restriction homomorphism $\mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{h}]$ restricts to $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$.

Recall (Chevalley - Shephard - Todd Thm from Sec 2.2 of Lec 9) that $\mathfrak{h}/W$ is smooth. $\mathbb{C}[\mathfrak{h}]^W$ is positively graded, so $\mathbb{C}[\mathfrak{h}]^W$ is the algebra of polynomials in $\operatorname{rk} \mathfrak{g}$ homogeneous elements, say $f_1, \ldots f_r$.

2.1) Deformations of $\mathbb{C}[\mathcal{N}]$.

Proposition: The following are true:

(i) $\mathcal{N} = \pi_G^{-1}(0)$ as subsets of $\mathfrak{g}$.

ii) $f_1, \ldots, f_r \in \mathbb{C}[\mathfrak{g}]$ form a regular sequence (Sec 1.1 in Lec 9)

iii) $\mathbb{C}[\mathfrak{g}]$ is a free graded $\mathbb{C}[\mathfrak{g}]^G$-module.

iv) $\pi_G^{-1}(0)$ is reduced and normal as a scheme.

Sketch of proof:

(i): • $\mathcal{N} \subset \pi_G^{-1}(0)$: in the proof in Sec 2.2 of Lec 7 (after Exer 7) we've see that $te \in Ge \Rightarrow \pi_G(te) = \pi_G(e)$ $\forall\, t \in \mathbb{C}^\times \Rightarrow \pi_G(e) = 0$.

• $\pi_G^{-1}(0) \subset \mathcal{N}$: let $\rho: \mathfrak{g} \to \mathfrak{sl}(V)$ be a faithful representation. Let $\chi_{V,2}, \dots, \chi_{V,k}$ be the coeff's of the char. polynomial of $\rho(x)$, $x \in \mathfrak{g}$. Then $\chi_{V,i} \in$ max. ideal of 0 in $\mathbb{C}[\mathfrak{g}]^G$. So $\chi_{V,i}(x) = 0$ for $x \in \pi_G^{-1}(0)$. Hence x acts on V by a nilpotent operator $\Leftrightarrow x \in \mathcal{N}$.

(ii) Follows from (i) & $\operatorname{codim}_{\mathfrak{g}} \pi_G^{-1}(0) = \operatorname{codim}_{\mathfrak{g}} \mathcal{N} = r$.

(iii) We'll use the following important fact (vanishing of the 1st Koszul homology):

Fact ([E], Cor. 17.5): Let R be a Noetherian commutative ring. Suppose $f_1, \dots, f_k \in R$ is a regular sequence, and $b_1, \dots, b_k \in R$ are s.t. $\sum_{i=1}^{k} f_i b_i = 0$. Then $\exists\, b_{ij} \in R$ w. $b_{ii} = 0$, $b_{ij} = -b_{ji}$ s.t. $b_i = \sum_{j=1}^{k} b_{ij} f_j$.

Now we get back to (iii). The algebra $\mathbb{C}[\pi_G^{-1}(0)]$ is graded. Pick a homogeneous vector space basis $\underline{b}_i \in \mathbb{C}[\pi_G^{-1}(0)]$, $i \in I$, & lift it to homogeneous $b_i \in \mathbb{C}[\mathfrak{g}]$. By the graded Nakayama lemma, b_i $(i \in I)$ span the $\mathbb{C}[\mathfrak{g}]^G$-module $\mathbb{C}[\mathfrak{g}]$, while Fact implies they are linearly independent (exercise).

iv) By (ii), $\pi_G^{-1}(0)$ is a complete intersection. By Serre's normality criterium ([E], Thm 18.5) we need to show that $\{x \in \mathcal{N} \mid d_x\pi_G \text{ is not surjective}\}$ has codim ≥ 2 in $\mathcal{N}$. Kostant proved, [Ko], that $d_x\pi_G$ is surjective $\forall\, x \in \mathbb{O}_{pr}$. Now we use that $\operatorname{codim}_{\mathcal{N}} \mathcal{N} \setminus \mathbb{O}_{pr} \geq 2$ and finish the proof. □

Corollary: For $a \in \mathfrak{h}/W$ consider the filtration on $\mathbb{C}[\pi_G^{-1}(a)]$ induced by the grading on $\mathbb{C}[\mathfrak{g}]$. Then we have a natural isomorphism $\mathbb{C}[\pi_G^{-1}(0)] \xrightarrow{\sim} \operatorname{gr} \mathbb{C}[\pi_G^{-1}(a)]$.

Proof: Set $a_i = f_i(a)$, $i = 1, \ldots, r$. Then $\mathbb{C}[\pi_G^{-1}(a)] = \mathbb{C}[\mathfrak{g}]/(f_i - a_i)_{i=1}^r$, $\mathbb{C}[\pi_G^{-1}(0)] = \mathbb{C}[\mathfrak{g}]/(f_i)$. We have the natural graded epimorphism $\mathbb{C}[\mathfrak{g}] \twoheadrightarrow \operatorname{gr} \mathbb{C}[\pi_G^{-1}(a)]$ that sends the f_i's to 0 so factors through $\mathbb{C}[\pi_G^{-1}(0)] \xrightarrow{(*)} \operatorname{gr} \mathbb{C}[\pi_G^{-1}(a)]$. It sends the $\underline{b}_i$'s from the proof of (iii) in Proposition to the image of the b_i's in $\mathbb{C}[\pi_G^{-1}(a)]$. Since b_i's form a basis in the $\mathbb{C}[\mathfrak{g}]^G$-module $\mathbb{C}[\mathfrak{g}]$, their images in $\mathbb{C}[\pi_G^{-1}(a)]$ form a basis, so $(*)$ in an iso. □

2.2) Poisson structures.

Recall that:

(I) $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ has the unique Poisson structure w. $\{\xi,\eta\} = [\xi,\eta]$ $\forall\, \xi,\eta\in\mathfrak{g}$.

(II) If X is a Poisson variety & $G \curvearrowright X$ is a Hamiltonian action, then the comoment map $\xi \mapsto H_\xi$ satisfies $H_{[\xi,\eta]} = \{H_\xi, H_\eta\}$. So it extends to a Poisson algebra homomorphism $\mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[X]$.

Definition: The center of the Poisson algebra A is $\{z\in A \mid \{z,a\}=0\ \forall\, a\in A\}$. It's a subalgebra.

Exercise 1: The center of $\mathbb{C}[\mathfrak{g}^*]$ is $\mathbb{C}[\mathfrak{g}^*]^G$ (hint: the former is $\{z\in \mathbb{C}[\mathfrak{g}^*] \mid \{\xi,z\}=0\ \forall\, \xi\in\mathfrak{g}\}$.

Now apply (II) to $X=\mathbb{O}_{pr}$. The moment map $\mu: \mathbb{O}_{pr}\to\mathfrak{g}^*$ $(\simeq\mathfrak{g})$ factors as $\mathbb{O}_{pr}\hookrightarrow\overline{\mathbb{O}}_{pr} = \mathcal{N}\hookrightarrow\mathfrak{g}^*$. Since $\mathbb{C}[\mathbb{O}_{pr}] = \mathbb{C}[\overline{\mathbb{O}}_{pr}]$, Rem 1 in Sec 1, we see that the epimorphism

$$\mathbb{C}[\mathfrak{g}^*] \twoheadrightarrow \mathbb{C}[\pi_G^{-1}(0)] = \mathbb{C}[\mathbb{O}_{pr}]$$

is Poisson.

Exercise 2: Let A be a Poisson algebra, Z be its center, & $I \subset Z$ an ideal. Then A/AI carries the unique Poisson bracket s.t. $A \twoheadrightarrow A/AI$ is a Poisson algebra homomorphism.

Applying this to $A = \mathbb{C}[\mathfrak{g}^*]$ & the maximal ideal $I \subset \mathbb{C}[\mathfrak{g}^*]^G$ of $a \in \mathfrak{g}^*/\!/G$ we get a Poisson bracket on $\mathbb{C}[\mathfrak{g}^*]/\mathbb{C}[\mathfrak{g}^*]I = \mathbb{C}[\pi_G^{-1}(a)]$

Exercise 3: 1) This Poisson bracket on $\mathbb{C}[\pi_G^{-1}(a)]$ has deg $\leqslant -1$ w.r.t. the filtration on $\mathbb{C}[\pi_G^{-1}(a)]$.

2) The isomorphism $\mathbb{C}[\pi_G^{-1}(0)] \xrightarrow{\sim} \operatorname{gr} \mathbb{C}[\pi_G^{-1}(a)]$ is Poisson.

So we have constructed a family of filtered Poisson deformations of $\mathbb{C}[\mathcal{N}]$ parameterized by points of $\mathfrak{g}/\!/G \simeq \mathfrak{h}/W$.

The following exercise examines the structure on $\pi_G^{-1}(a)$.

Exercise 4: 1) The unique closed G-orbit in $\pi_G^{-1}(a)$ is s/simple.

2) $\pi_G^{-1}(a)$ contains the unique open orbit, say $\mathbb{O}_{pr,a}$

3) $\mathbb{C}[\pi_G^{-1}(a)] \xrightarrow{\sim} \mathbb{C}[\mathbb{O}_{pr,a}]$.