Lecture 11.

1) Filtered quantizations of $\mathbb{C}[\mathcal{N}]$.

2) Main classification result.

3) Namikawa-Cartan space & construction of quantizations.

Ref: [L1], [N2a], [N2b]

1) Let G be a semisimple algebraic group & $\mathcal{N}\subset\mathfrak{g}^*(\simeq\mathfrak{g})$ be the nilpotent cone. In Lecture 11 we have constructed a family of filtered Poisson deformations of $\mathbb{C}[\mathcal{N}]$ parameterized by points of $\mathfrak{g}^{(*)}/\!\!/G \xrightarrow{\sim} \mathfrak{h}^{(*)}/W$. More precisely, $\mathbb{C}[\mathfrak{g}^*]^G$ is the Poisson center of $\mathbb{C}[\mathfrak{g}^*]$. For $\lambda\in\mathfrak{g}^*/\!\!/G$, let $\mathfrak{m}_\lambda\subset\mathbb{C}[\mathfrak{g}^*]^G$ be its maximal ideal. Then we can form the filtered Poisson deformation $\mathcal{A}^\circ_\lambda := \mathbb{C}[\mathfrak{g}^*]/\mathbb{C}[\mathfrak{g}^*]\mathfrak{m}_\lambda$ of $\mathbb{C}[\mathcal{N}]$.

Recall that the center Z of $U(\mathfrak{g})$ is also identified w. $\mathbb{C}[\mathfrak{h}^*]^W$ (the Harish-Chandra isomorphism). So to $\lambda\in\mathfrak{h}^*/W$ we can assign the algebra $\mathcal{U}_\lambda := U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\lambda$. It inherits the filtration from the PBW filtration of $U(\mathfrak{g})$, & we have $\deg[\cdot,\cdot]\leqslant -1 \rightsquigarrow \deg -1$ bracket on $\operatorname{gr}\mathcal{U}_\lambda$.

Theorem: $\mathcal{U}_\lambda$ is a filtered quantization of $\mathbb{C}[\mathcal{N}]$ (w. grading induced from the usual grading on $S(\mathfrak{g})$).

Proof:

Step 1: Here we are going to establish a graded Poisson algebra epimorphism $\mathbb{C}[\mathcal{N}] \twoheadrightarrow \mathrm{gr}\,\mathcal{U}_\lambda$. Recall free homogeneous generators $f_1,\dots,f_r$ ($r=\mathrm{rk}\,\mathfrak{g}$) of $\mathbb{C}[\mathfrak{g}]^G=\mathbb{C}[\mathfrak{h}]^W$. We can view them as elements of Z, in this case we write $\hat{f}_1,\dots,\hat{f}_r$. So $\mathcal{U}_\lambda$ takes the form $\mathcal{U}_\lambda=\mathcal{U}(\mathfrak{g})/(\hat{f}_i-a_i)_{i=1}^r$ for $a_i\in\mathbb{C}$. Under the isomorphism $\mathrm{gr}\,\mathcal{U}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{g})$, the class $\hat{f}_i-a_i+\mathcal{U}(\mathfrak{g})_{<d_i}$ ($d_i:=\deg f_i$) is f_i. So $(f_1,\dots,f_r)\subset \mathrm{gr}((\hat{f}_i-a_i)) \rightsquigarrow$ graded algebra epimorphism

$$\mathbb{C}[\mathcal{N}]=\mathbb{C}[\mathfrak{g}^*]/(f_1,\dots,f_r) \twoheadrightarrow [\mathrm{gr}\,\mathcal{U}(\mathfrak{g})]/[\mathrm{gr}((\hat{f}_i-a_i))]=\mathrm{gr}\,\mathcal{U}_\lambda$$

It's Poisson: we have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{g}^*] \arrow[r, "\sim", "2"'] \arrow[d, two heads, "1"'] & \mathrm{gr}\,\mathcal{U}(\mathfrak{g}) \arrow[d, two heads, "3"] \\
\mathbb{C}[\mathcal{N}] \arrow[r, two heads] & \mathrm{gr}\,\mathcal{U}_\lambda
\end{tikzcd}$$

where arrows 1, 2, 3 are Poisson (details are exercise).

Step 2: It remains to show that $\mathbb{C}[\mathcal{N}] \twoheadrightarrow \mathrm{gr}\,\mathcal{U}_\lambda$ is iso.

Note that the elements $\hat{f}_1, \dots \hat{f}_r$ pairwise commute. So our claim is a consequence of the following.

Claim: Let A be a $\mathbb{Z}_{\geq 0}$-graded Noetherian Poisson $\mathbb{C}$-algebra w. $\deg\{\cdot,\cdot\} = -d < 0$. Let $\mathcal{A}$ be its filtered quantization. Let $\hat{f}_i \in \mathcal{A}_{\leq d_i}$ & $f_i := \hat{f}_i + \mathcal{A}_{<d_i} \in A_{d_i}$, $i = 1, \dots r$, $d_i \in \mathbb{Z}_{>0}$. Suppose that:

(a) $f_1, \dots f_r \in A$ form a regular sequence.

(b) $[\hat{f}_j, \hat{f}_k] = \sum_{i=1}^{r} \hat{c}^i_{jk} \hat{f}_i$ w. $\hat{c}^k_{ij} \in \mathcal{A}_{\underbrace{\leq d_i + d_j - d - d_k}_{\text{expected degree}}}$

Then $\operatorname{gr} \mathcal{A}(\hat{f}_1, \dots \hat{f}_r) = A(f_1, \dots f_r)$.

Proof of Claim: Assume the contrary: $\exists\, e > 0$ & $\hat{b}_i \in \mathcal{A}_{\leq e - d_i}$ w. $b_i := \hat{b}_i + \mathcal{A}_{<e-d_i} (\in A_{e-d_i})\,|$ the top degree (nonzero) term in $\sum_{i=1}^{r} \hat{b}_i \hat{f}_i$ is not in $A(f_1, \dots, f_r)$. Assume e is minimal so that this happens. The deg e term in $\sum_{i=1}^{r} \hat{b}_i \hat{f}_i$ is $\sum_{i=1}^{r} b_i f_i$ – it must be 0: $\sum_{i=1}^{r} b_i f_i$ is in $A(f_1, \dots f_r)$, if it's nonzero, it is the top deg nonzero term of $\sum_{i=1}^{r} \hat{b}_i \hat{f}_i$.

Using (a) & Fact in Sec 2.1 of Lec 10 we see $\exists\, b_{ij} \in A$ s.t. $b_{ij} = -b_{ji}$ & $b_i = \sum_{j=1}^{r} b_{ij} f_j$. We can assume that $b_{ij} \in A_{e - d_i - d_j}$. Lift them to $\hat{b}_{ij} \in \mathcal{A}_{\leq e - d_i - d_j}$ w. $\hat{b}_{ij} = -\hat{b}_{ji}$. Then $\hat{b}'_i := \hat{b}_i - \sum_{j=1}^{r} \hat{b}_{ij} \hat{f}_j \in \mathcal{A}_{< e - d_i}$. We have:

$$\sum_{i=1}^{r} \hat{b}_i \hat{f}_i = \sum_{i=1}^{r} \hat{b}_i' \hat{f}_i + \sum_{i,j=1} \hat{b}_{ij} \hat{f}_i \hat{f}_j = [\hat{b}_{ij} = -\hat{b}_{ji}] = \sum_{i=1}^{r} \hat{b}_i' \hat{f}_i$$

$$+ \sum_{j<k} \hat{b}_{jk} [\hat{f}_j, \hat{f}_k] = [(6)] = \sum_{i=1}^{r} (\underbrace{\hat{b}_i' + \sum_{j<k} \hat{b}_{jk} \hat{c}^{i}_{jk}}_{\in \mathcal{A}_{<e-d_i}}) \hat{f}_i$$

This contradicts the minimality of e & finishes the proof $\square$

Rem: Both filtered Poisson deformations (Sec 2 of Lec 10) & filtered quantizations of $\mathbb{C}[\mathcal{N}]$ we have constructed are specializations of the same algebra: the $\mathbb{C}[\mathfrak{h}/W][\hbar]$-algebra $R_\hbar(\mathcal{U})$: for Poisson deformations we look at specializations at $\hbar=0$, and for quantizations - at $\hbar=1$.

2) Main classification result.

Let X be a conical symplectic singularity. The following theorem is the main result of the 1st (larger) part of the course.

Theorem: There is a finite dimensional Namikawa-Cartan space, and a finite reflection group $W_X \subset GL(\mathfrak{h}_X)$, Namikawa-Weyl group,

s.t. $\mathfrak{h}_X/W_X$ (an affine space!) is in natural bijections w.
- {filtered Poisson deformations of $\mathbb{C}[X]$}/iso ([N2a], [N2b])
- {filtered quantizations of $\mathbb{C}[X]$}/iso ([L1]).

Example: Let $X=\mathcal{N}$. Then $\mathfrak{h}_X=\mathfrak{h}^*$ ($\simeq\mathfrak{h}$ - the canonical ident'n is with $\mathfrak{h}^*$) & $W_X=W$. In fact, the above construction yields the classification.

Remark: As in the case of $X=\mathcal{N}$, the filtered Poisson deformations & filtered quantizations are specializations (at $\hbar=0$ & at $\hbar=1$) of a "universal deformation", a graded $\mathbb{C}[\mathfrak{h}_X/W_X][\hbar]$-algebra free over $\mathbb{C}[\mathfrak{h}_X/W_X][\hbar]$ and specializing to $\mathbb{C}[X]$ at $(0,0)\in\mathfrak{h}/W\times\mathbb{C}$.

3) Namikawa-Cartan space and construction of quantizations.

We will start our (long) discussion of Theorem from Sec 2 by explaining the geometric meaning of $\mathfrak{h}_X$ (part of a reason, $\mathfrak{h}_X$ & $\mathfrak{h}_X/W_X$ are isomorphic affine spaces).

3.1) The case when X has a symplectic resolution.

Suppose $\pi: Y \longrightarrow X$ is a symplectic resolution.

Fact (to be elaborated later in the course): $\mathfrak{h}_X \xrightarrow{\sim} H^2(Y,\mathbb{C})$.

Example: $X=\mathcal{N}$. Then for Y we take $T^*(G/B)\simeq G\times^B\mathfrak{n}$. Then we have $H^2(T^*(G/B),\mathbb{C})\xrightarrow{\sim} H^2(G/B,\mathbb{C})$. The latter is identified with $\mathfrak{h}^*$. For example, G/B has a stratification by affine spaces (the Schubert stratification). It follows that $H^{odd}(G/B,\mathbb{Z})=\{0\}$, while $H^{2i}(G/B,\mathbb{Z})$ is a free abelian group whose basis is labelled by the codim i strata. For $i=1$, the codim 2 strata are labelled by the simple reflections, the corresponding basis element in $H^2(G/B,\mathbb{C})$ corresponds to the simple root in $\mathfrak{h}^*$.

Let's explain what role a symplectic resolution plays. Recall that X has rational singularities (Sec 1.3 of Lec 9). So we have $R^i\pi_*\mathcal{O}_Y=0$ for $i>0$ (and $\pi_*\mathcal{O}_Y\xrightarrow{\sim}\mathcal{O}_X$). The usefulness of this is 2-fold (we'll elaborate on this later in the course).

- One can talk about quantizations of $\mathcal{O}_Y$ - these are certain sheaves of algebras on Y. Since Y is smooth & symplectic & $H^i(Y,\mathcal{O}_Y)=0$, $i=1,2$, (b/c $R^i\pi_*\mathcal{O}_Y=0$ & X is affine), one can show that the quantizations of $\mathcal{O}_Y$ are classified by $H^2(Y,\mathbb{C})$.

- For $\lambda\in\mathfrak{h}_X=H^2(Y,\mathbb{C})$, let $\mathcal{D}_\lambda$ denote the corresponding quantization of $\mathcal{O}_Y$. Thx to $H^1(Y,\mathcal{O}_Y)=0$, $\Gamma(\mathcal{D}_\lambda)$ is a quantization of $\mathbb{C}[Y]=\mathbb{C}[X]$. This turns out to exhaust all quantizations of $\mathbb{C}[X]$ and $\Gamma(\mathcal{D}_\lambda)\simeq\Gamma(\mathcal{D}_{\lambda'})$ as quantizations $\Leftrightarrow$ $\lambda'\in W_X\lambda$.

3.2) General case

In general, a conical symplectic singularity X doesn't admit a symplectic resolution. Our first attempt could be to take <u>some</u> resolution $\pi: Y\to X$ and try to argue as in the previous section. This doesn't work: to talk about quantizations of Y, Y needs to be Poisson, and in general it's not

On the other hand we can talk about <u>partial</u> Poisson resolutions.

Definition: Let Y be a normal Poisson variety. A morphism $\pi: Y\to X$ is called a partial Poisson resolution if:

• it's proper & birational (note that we do not require Y to be smooth so π is only a partial resolution).

• it's Poisson (meaning that for open affines $X' \subset X$, $Y' \subset Y$ w. $\pi(Y') \subset X'$ the homomorphism $\pi^*: \mathbb{C}[X'] \to \mathbb{C}[Y']$ is Poisson).

Here's an important property of Y.

Proposition: Y is singular symplectic.

Proof: We need to show that Y^{reg} is symplectic & the symplectic form extends to a resolution of Y.

Let P_Y, P_X denote the Poisson bivectors of X^{reg} & Y^{reg}. Let $Y^\circ \subset Y^{reg}$ be open s.t. $\pi: Y^\circ \to X$ is an open embedding. So $\pi^*(P_X|_{\pi(Y^\circ)}) = P_Y|_{Y^\circ}$ b/c π is Poisson.

On the other hand, we can find a resolution $\rho: Z \to Y$ that is iso over Y^{reg}. Since X is singular symplectic, for the symplectic form ω_X^{reg} on X^{reg}, we know that $(\rho \circ \pi)^* \omega_{X^{reg}}$ extends to Z. Hence $\pi^*(\omega_X^{reg}|_{\pi(Y^\circ)})$ extends to a regular 2-form on Y^{reg}.

Note that ω_X^{reg} & $P_X|_{X^{reg}}$ are inverse to each other. Being inverse is a polynomial condition. So the extensions, $\mathcal{P}_Y|_{Y^{reg}}$ of $\mathcal{P}_Y|_{Y^\circ}$, and $(\rho\circ\pi)^*\omega_{X^{reg}}|_{Y^{reg}}$ of $\pi^*(\omega_X^{reg}|_{\pi(Y^\circ)})$, are inverse to each other.

Hence Y^{reg} is symplectic, let ω_Y^{reg} denote the symplectic form. Note that $(\rho\circ\pi)^*\omega_X^{reg}$ & $\rho^*\omega_Y^{reg}$ coincide on $\rho^{-1}(Y^\circ)$. So the 2-form on Z extending the former also extends the latter. $\square$

Exercise: π is an isomorphism over X^{reg}.

The variety Y is still singular, so one cannot classify its quantizations directly. However, it turns out that one can choose Y to be maximal (meaning that there are no nontrivial partial Poisson resolutions to Y). This is a nontrivial result. Moreover, for such Y we have $\operatorname{codim}_Y Y\setminus Y^{reg} \geqslant 4$, from which one can deduce that $\mathbb{C}[Y^{reg}] \xleftarrow{\sim} \mathbb{C}[Y] \xleftarrow{\sim} \mathbb{C}[X]$ & $H^i(Y^{reg}, \mathcal{O}_Y) = 0$ for $i=1,2$. Then one can apply the classification of quantizations to Y^{reg} instead of Y, they are parameterized by $H^2(Y^{reg}, \mathbb{C})$. And so we set $\mathfrak{h}_X = H^2(Y^{reg}, \mathbb{C})$.

We'll elaborate on the algebraic geometry of Y in the next lecture & on classification of deformations later.