Lecture 11

1) Filtered quantizations of C[N]

2) Main classification result.

3) Namikawa-Cartan space & construction of quantizations.

Ref: [L1], [N2a], [N2b]

1) Let G be a semisimple algebraic group & N = g*(=g) be the nilpotent cone. In Lecture 11 we have constructed a family of filtered Poisson deformations of C[N] parameterized by points of g*/W. More precisely, C[g*]^g is the Poisson center of C[g*]. For λ ∈ g*/W, let m_λ ⊂ C[g*]^g be its maximal ideal. Then we can form the filtered Poisson deformation H_λ := C[g*]/C[g*]m_λ of C[N].

Recall that the center Z of U(g) is also identified w. C[g]^W (the Harish-Chandra isomorphism). So to λ ∈ g*/W we can assign the algebra U_λ := U(g)/U(g)m_λ. It inherits the filtration from the PBW filtration of U(g), & we have deg [;] ≤ -1 for deg -1 bracket on gr U_λ.
Theorem: $U_\lambda$ is a filtered quantization of $C[N]$ (with grading induced from the usual grading on $S(g)$).

Proof:

Step 1: Here we are going to establish a graded Poisson algebra epimorphism $C[N] \twoheadrightarrow \text{gr } U_\lambda$. Recall free homogeneous generators $f_1, \ldots, f_r$ of $C[g]^\mathbb{C} = C[f]^\mathbb{W}$. We can view them as elements of $\mathbb{Z}$; in this case we write $\hat{f}_1, \ldots, \hat{f}_r$. So $U_\lambda$ takes the form

$U_\lambda = U(\mathfrak{g})/(\hat{f}_i-a_i), \text{ for } a_i \in \mathbb{C}$. Under the isomorphism $\text{gr } U(\mathfrak{g}) \twoheadrightarrow S(\mathfrak{g})$, the class $\hat{f}_i - a_i + U(\mathfrak{g}) e_i$ ($e_i = \text{deg } f_i$) is $f_i$. So $(\hat{f}_1, \ldots, \hat{f}_r) \in \text{gr } ((\hat{f}_i - a_i)) \twoheadrightarrow$ graded algebra epimorphism

$C[N] = C[\mathfrak{g}^*]/(f_1, \ldots, f_r) \twoheadrightarrow [\text{gr } U(\mathfrak{g})]/[\text{gr } ((\hat{f}_i - a_i))] = \text{gr } U_\lambda$

It's Poisson: we have a commutative diagram

$$
\begin{array}{ccc}
C[\mathfrak{g}^*] & \xrightarrow{\sim} & \text{gr } U(\mathfrak{g}) \\
\downarrow^{\text{1}} & & \downarrow^{\text{3}} \\
C[N] & \rightarrow & \text{gr } U_\lambda
\end{array}
$$

where arrows 1, 2, 3 are Poisson (details are exercise).

Step 2: It remains to show that $C[N] \twoheadrightarrow \text{gr } U_\lambda$ is iso.
Note that the elements $\hat{f}_1, \ldots, \hat{f}_r$ pairwise commute. So our claim is a consequence of the following.

**Claim:** Let $A$ be a $\mathbb{Z}_{\geq 0}$-graded Noetherian Poisson $C$-algebra w. $\deg \xi_i = -d < 0$. Let $\mathcal{A}$ be its filtered quantization. Let $\hat{f}_i \in \mathcal{A}_{e - d_i}$ & $\hat{f}_i = \hat{f}_i + \mathcal{A}_{e - d_i} \in A_{d_i}$, $i = 1, \ldots, r$, $d_i \in \mathbb{Z}_{\geq 0}$. Suppose that:

1. $\hat{f}_1, \ldots, \hat{f}_r \in A$ form a regular sequence.
2. $[\hat{f}_j, \hat{f}_k] = \sum_{i=1}^{r} \hat{c}^i_{jk} \hat{f}_i$ w. $\hat{c}^i_{jk} \in \mathcal{A}_{e - d_i + d_j - d_k}$, $\deg \hat{c}^i_{jk} = \deg(\hat{f}_i)$.

Then $\text{gr} \mathcal{A}(\hat{f}_1, \ldots, \hat{f}_r) = A(\hat{f}_1, \ldots, \hat{f}_r)$.

**Proof of Claim:** Assume the contrary: $\exists$ $e \geq 0$ & $\hat{b}_i \in \mathcal{A}_{e - d_i}$ w. $\hat{b}_i = \hat{b}_i + \mathcal{A}_{e - d_i} (\in A_{e - d_i})$ (the top degree (nonzero) term in $\sum_{i=1}^{r} \hat{b}_i$ is not in $A(\hat{f}_1, \ldots, \hat{f}_r)$). Assume $e$ is minimal so that this happens. The degree term in $\sum_{i=1}^{r} \hat{b}_i$, is $\sum_{i=1}^{r} \hat{b}_i \hat{f}_i$ — it must be $0$: $\sum_{i=1}^{r} \hat{b}_i \hat{f}_i$ is in $A(\hat{f}_1, \ldots, \hat{f}_r)$, if it's nonzero, it is the top degree nonzero term of $\sum_{i=1}^{r} \hat{b}_i \hat{f}_i$.

Using (a) & Fact in Sec 2.1 of Lec 10, we see $\exists \hat{b}_{ij} \in A$ s.t. $\hat{b}_{ij} = -\hat{b}_{ji}$ & $\hat{b}_i = \sum_{j=1}^{r} \hat{b}_{ij} \hat{f}_j$. We can assume that $\hat{b}_{ij} \in \mathcal{A}_{e - d_i - d_j}$.

Lift them to $\hat{b}_{ij} \in \mathcal{A}_{e - d_i - d_j}$ w. $\hat{b}_{ij} = -\hat{b}_{ji}$. Then $\hat{b}_i' = \hat{b}_i - \sum_{j=1}^{r} \hat{b}_{ij} \hat{f}_j \in \mathcal{A}_{e - d_i}$. We have:
\[
\sum_{i=1}^{\xi} \hat{b}_i \hat{f}_i = \sum_{i=1}^{\xi} \hat{b}_i \hat{f}_i + \sum_{\substack{j<k \in \mathcal{A} \setminus \mathcal{E} \setminus \mathcal{D}_i}} b_{jk} \left[ \hat{f}_j, \hat{f}_k \right] = [b_{ij} - b_{ji}] = \sum_{i=1}^{\xi} \hat{b}_i \hat{f}_i
\]

\[
+ \sum_{\substack{j<k \in \mathcal{A} \setminus \mathcal{E} \setminus \mathcal{D}_i}} b_{jk} \left[ \hat{f}_j, \hat{f}_k \right] = [b_{jk}] = \sum_{i=1}^{\xi} \left( \hat{b}_i + \sum_{\substack{j<k \in \mathcal{A} \setminus \mathcal{E} \setminus \mathcal{D}_i}} b_{jk} \hat{c}_i \right) \hat{f}_i
\]

This contradicts the minimality of \( e \) & finishes the proof. \( \square \)

Rem: Both filtered Poisson deformations (Sec 2 of Lec 10) & filtered quantizations of \( \mathbb{C}[N] \) we have constructed are specializations of the same algebra: the \( \mathbb{C}[\hbar \mathbb{W}] \)-algebra \( R_\hbar(U) \): for Poisson deformations we look at specializations at \( \hbar = 0 \), and for quantizations - at \( \hbar = 1 \).

2) Main classification result.

Let \( X \) be a conical symplectic singularity. The following theorem is the main result of the 1st (larger) part of the course.

Theorem: There is a finite dimensional Namikawa-Cartan space, and a finite reflection group \( W_x \subset GL(\mathfrak{g}_x) \), Namikawa-Weyl group,
s.t. \( \mathfrak{g}_x/W_x \) (an affine space!) is in natural bijections w.

- filtered Poisson deformations of \( \mathbb{C}[X]/\text{iso} \) ([N2a], [N2b])
- filtered quantizations of \( \mathbb{C}[X]/\text{iso} \) ([11]).

**Example:** Let \( X = \mathbb{N} \). Then \( \mathfrak{g}_x = \mathfrak{g}^\ast(\cong \mathfrak{g} \text{ - the canonical identification is with } \mathfrak{g}^\ast) \) & \( W_x = W \). In fact, the above construction yields the classification.

**Remark:** As in the case of \( X = \mathbb{N} \), the filtered Poisson deformations & filtered quantizations are specializations (at \( \hbar = 0 \) & at \( \hbar = 1 \)) of a "universal deformation", a graded \( \mathbb{C}[\mathfrak{g}_x/W_x][\hbar] \)-algebra free over \( \mathbb{C}[\mathfrak{g}_x/W_x][\hbar] \) and specializing to \( \mathbb{C}[X] \) at \( (q, 0) \in \mathfrak{g}/W \times \mathbb{C} \).

3) **Namikawa-Cartan space and construction of quantizations.**

We will start our (long) discussion of Theorem from Sec 2 by explaining the geometric meaning of \( \mathfrak{g}_x \) (part of a reason, \( \mathfrak{g}_x \& \mathfrak{g}_x/W_x \) are isomorphic affine spaces).
3.1) The case when $X$ has a symplectic resolution

Suppose $\sigma: Y \rightarrow X$ is a symplectic resolution.

**Fact** (to be elaborated later in the course): $\mathfrak{k}_X \cong H^2(Y, \mathbb{C})$.

**Example**: $X = N$. Then for $Y$ we take $T^*(G/B) \cong \mathbb{C} \times \mathfrak{k}$. Then we have $H^2(T^*(G/B), \mathbb{C}) \cong H^2(G/B, \mathbb{C})$. The latter is identified with $\mathfrak{k}_X^*$. For example, $G/B$ has a stratification by affine spaces (the Schubert stratification). It follows that $H^{2*}(G/B, \mathbb{Z}) = \mathfrak{k}_X^*$, while $H^{2*}(G/B, \mathbb{Z})$ is a free abelian group whose basis is labelled by the codim $i$ strata. For $i=1$, the codim 2 strata are labelled by the simple reflections, the corresponding basis element in $H^2(G/B, \mathbb{C})$ corresponds to the simple root in $\mathfrak{k}_X^*$.

Let's explain what role a symplectic resolution plays. Recall that $X$ has rational singularities (Sec 1.3 of Lec 9). So we have $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$ (and $\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$). The usefulness of this is 2-fold (we'll elaborate on this later in the course):

- One can talk about quantizations of $\mathcal{O}_Y$—these are certain
Sheaves of algebras on $Y$. Since $Y$ is smooth & symplectic & $H^i(Y, O_Y) = 0$, $i = 1, 2$, ($b_c R^i f_* Q_Y = 0$ & $X$ is affine), one can show that the quantizations of $O_Y$ are classified by $H^2(Y, C)$.

- For $\lambda \in Y = H^2(Y, C)$, let $D_{\lambda}$ denote the corresponding quantization of $O_Y$. Thus, to $H^2(Y, O_Y) = 0$, $\Gamma(D_{\lambda})$ is a quantization of $C[Y] = C[X]$. This turns out to exhaust all quantizations of $C[X]$ and $\Gamma(D_{\lambda}) \approx \Gamma(D_{\lambda'})$ as quantizations $\iff \lambda' \in \mathcal{H}_\lambda$.

### 3.2) General case

In general, a conical symplectic singularity $X$ doesn't admit a symplectic resolution. Our first attempt could be to take some resolution $\pi: Y \to X$ and try to argue as in the previous section. This doesn't work: to talk about quantizations of $Y$, $Y$ needs to be Poisson, and in general it's not. On the other hand, we can talk about partial Poisson resolutions.

**Definition:** Let $Y$ be a normal Poisson variety. A morphism $\pi: Y \to X$ is called a partial Poisson resolution if:
• it's proper & birational (note that we do not require Y to be smooth so π is only a partial resolution).
• it's Poisson (meaning that for open affines \( X' \subset X, \ Y' \subset Y \) w. π′(Y′) ⊂ X′, the homomorphism \( π^* : \mathbb{C}[X'] \rightarrow \mathbb{C}[Y'] \) is Poisson).

Here’s an important property of \( Y \).

**Proposition:** \( Y \) is singular symplectic.

**Proof:** We need to show that \( Y^{\text{reg}} \) is symplectic & the symplectic form extends to a resolution of \( Y \).

Let \( P_y, P_x \) denote the Poisson bivectors of \( X^{\text{reg}} \) & \( Y^{\text{reg}} \).

Let \( Y^0 \subset Y^{\text{reg}} \) be open s.t. \( π : Y^0 \rightarrow X \) is an open embedding.

So \( \pi^*(P_y|_{π(Y^0)}) = P_x|_{π(Y^0)} \) b/c \( π \) is Poisson.

On the other hand, we can find a resolution \( π : Z \rightarrow Y \) that is iso over \( Y^{\text{reg}} \). Since \( X \) is singular symplectic, for the symplectic form \( \omega_X^{\text{reg}} \) on \( X^{\text{reg}} \), we know that \((p \circ π)^* \omega_X^{\text{reg}}\) extends to \( Z \). Hence \( π^*(\omega_X^{\text{reg}}|_{π(Y^0)}) \) extends to a regular \( Z \)-form on \( Y^{\text{reg}} \).

Note that \( \omega_X^{\text{reg}} \) & \( P_x|_{X^{\text{reg}}} \) are inverse to each other. Being
inverse is a polynomial condition. So the extensions, $P_y|_{y^*}$ of $P_y|_{y^*}$, and $(p \circ \pi)^* \omega_{x|y^*}|_{y^*}$ of $\pi^* (\omega_{x|y^*}|_{y^*})$, are inverse to each other.

Hence $Y^*_{x|y^*}$ is symplectic, let $\omega_{x|y^*}$ denote the symplectic form. Note that $(p \circ \pi)^* \omega_{x|y^*}^x$ & $p^* \omega_{x|y^*}^x$ coincide on $p^{-1}(Y^*)$. So the 2-form on $Z$ extending the former also extends the latter. $\square$

Exercise: $\pi$ is an isomorphism over $X^*_{x|y^*}$

The variety $Y$ is still singular, so one cannot classify its quantizations directly. However, it turns out that one can choose $Y$ to be maximal (meaning that there are no nontrivial partial Poisson resolutions to $Y$). This is a nontrivial result. Moreover, for such $Y$ we have $\text{codim}_Y Y^*_{y^*} \geq 4$, from which one can deduce that $C[Y^*_{y^*}] \leftarrow C[Y] \leftarrow C[X]$ & $H^i(Y^*_{y^*}, \mathcal{O}_Y) = 0$ for $i=1,2$. Then one can apply the classification of quantizations to $Y^*_{y^*}$ instead of $Y$, they are parameterized by $H^2(Y^*_{y^*}, C)$. And so we set $k_x = H^2(Y^*_{y^*}, C)$. 

9
We'll elaborate on the algebraic geometry of $Y$ in the next lecture & on classification of deformations later.