Lecture 12

1) $\mathbb{Q}$-factorial terminalizations.

2*) Cohomology vanishing.

Refs: [KP1]; [KP2]; Sec 6 in [CM], Sec 7 in [D]; [BCHM], [L3].

1) In Sec 3.2 of Lec 11, we have discussed partial Poisson resolutions of a conical symplectic singularity X. We've stated that there's a maximal such partial resolution Y. In this lecture we'll investigate the geometry of such Y. We'll see, in particular, that Y is "terminal" & "$\mathbb{Q}$-factorial", two kinds of singularities that are relevant for the MMP (minimal model program) in Birational Geometry.

1.1) Terminal varieties.

We are only going to discuss this in the context of symplectic singularities: strictly speaking the following definition is a result of Namikawa.

Definition: Let Y be a singular symplectic variety. We say Y is *terminal* if $\operatorname{codim}_Y (Y\setminus Y^{reg})\geqslant 4$.

Consider a special case: let $\mathbb{O}\subset\mathfrak{g}^*$ be a nilpotent orbit. Assume $\overline{\mathbb{O}}$ is normal. This is true for all $\mathbb{O}$ if $\mathfrak{g}=\mathfrak{sl}_n$ ([KP1]) but may fail for some $\mathbb{O}$ for other $\mathfrak{g}$ (and it's (mostly) known when this happens).

Lemma: TFAE:

a) $X=\overline{\mathbb{O}}$ is terminal.

b) $\operatorname{codim}_{\overline{\mathbb{O}}} \overline{\mathbb{O}}\setminus\mathbb{O}\geqslant 4$.

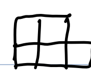
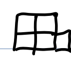
Proof:

Step 0: b) $\Rightarrow$ a) is easy as $\mathbb{O}\subset X^{reg}$. In what follows we prove $\mathbb{O}=X^{reg}$, which yields a) $\Rightarrow$ b).

Step 1: A closed subvariety Z in an affine Poisson variety X is called Poisson if $\{f,g\}\in I(Z)$, $\forall f\in\mathbb{C}[X], g\in I(Z)$, where $I(Z)$ is the ideal of Z. We claim that $Z\subset X=\overline{\mathbb{O}}$ is Poisson $\Leftrightarrow$ Z is G-stable. As we have argued several times (e.g. in the proof of Lemma in Sec 2.2 of Lec 7).

the restriction map $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \longrightarrow \mathbb{C}[X]$ is Poisson. Also, it's surjective. So, $\mathbb{C}[X]$ is generated by the image of $\mathfrak{g}$. Hence $I(Z)$ is Poisson iff it's $\mathfrak{g}$-stable $\Leftrightarrow$ G-stable.

Step 2: Now we prove that $\forall$ Poisson subvariety $Z \Rightarrow Z \cap \mathbb{O} = \emptyset$. This boils down to showing that a smooth symplectic (affine) variety Y has no proper Poisson subvarieties. This is because $\{f,g\} = v(f).g$ ($v(f)$ is a Hamiltonian vector field associated to f) & $\mathrm{Span}(v_y(f) \mid f \in \mathbb{C}[X]) = T_y Y$, $\forall y \in Y$. $\square$

Example: Let $\mathfrak{g} = \mathfrak{sl}_n$, let τ be a partition of n, and let $\mathbb{O}_\tau$ be the orbit w. Jordan type τ. We write $\tau^t = (\tau_1^t, \dots \tau_k^t)$ for the transpose of τ, e.g. $\tau =$  $\rightsquigarrow \tau^t =$ . Then $\dim \mathbb{O}_\tau = n^2 - 1 - \sum_{i=1}^{k} (\tau_i^t)^2$, [CM], Sec 6.1. Further, we have $\mathbb{O}_{\tau'} \subseteq \overline{\mathbb{O}_\tau}$ iff $\tau' \leq \tau$ in the standard "dominance order": $\sum_{i=1}^{\ell} \tau_i' \leq \sum_{i=1}^{\ell} \tau_i$, [CM], Sec 6.2. The following claim is a boring combinatorics exercise: for $\mathbb{O} := \mathbb{O}_\tau$, (6) of Lemma $\Leftrightarrow$ $\tau_i - \tau_{i+1} \in \{0,1\}$ $\forall i$. For example, for $\tau =$  this fails, so $\overline{\mathbb{O}_\tau}$ is not terminal, while for $\tau =$  it is.

Remark: While $\overline{\mathbb{O}}$ may fail to be normal in types B, C, D the conclusions of the previous example still hold (mostly thx to [KP2]) including the combinatorial criterium of being terminal in terms of partitions.

For the exceptional types the situation is more subtle: in a few cases one can have that $\operatorname{codim}_{\overline{\mathbb{O}}} \overline{\mathbb{O}} \setminus \mathbb{O} = 2$ but $\operatorname{codim}_X X \setminus X^{reg} \geqslant 4$ ($X = \operatorname{Spec} \mathbb{C}[\mathbb{O}]$).

One can also determine when $X = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$ is terminal.

1.2) $\mathbb{Q}$-factorial varieties.

Let X be a normal variety. To X we can assign two abelian groups. First, there's the Picard group, $\operatorname{Pic}(X)$, whose elements are isomorphism classes of line bundles and the multiplication comes from the tensor product of line bundles. The 2nd group is $\operatorname{Pic}(X^{reg})$ (identified w. the class group $Cl(X)$). See Sec 6 of Chapter 2 in Hartschorne's book.

Note that we have the restriction homomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(X^{reg})$. It's injective (exercise).

Definition: We say that X is *$\mathbb{Q}$-factorial* if the cokernel of the map $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{reg})$ is torsion.

Remark: the stronger condition that $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X^{reg})$ is equivalent to saying that X is *locally factorial* meaning that $\mathcal{O}_{X,x}$ is factorial (a.k.a. UFD) $\forall\, x \in X$.

We are going to analyze what $\mathbb{Q}$-factorial means when $X = \mathrm{Spec}\, \mathbb{C}[\widetilde{\mathbb{O}}]$ for equivariant covers $\widetilde{\mathbb{O}}$ of nilpotent orbits. We'll need the following construction from the theory of algebraic group actions. See [D], Sec 7, for details.

Definition: Let $\mathcal{L}$ be a line bundle on a scheme X. For an algebraic group action $H \curvearrowright X$, by an *H-linearization* of $\mathcal{L}$ we a lift of the H-action to the total space of $\mathcal{L}$ by fiberwise linear automorphisms. The group of isomorphism classes of H-linearized line bundles on X is denoted by $\mathrm{Pic}^H(X)$.

Theorem: Let X be a normal variety & H a connected algebraic group acting on X. Assume H is factorial (i.e $\mathbb{C}[H]$ is). Then every line bundle is H-linearizable.

Examples: 1) A torus, $(\mathbb{C}^\times)^n$, is factorial.

2) A s/simple group is factorial $\Leftrightarrow$ it's simply connected. This is proved using the Bruhat decomposition.

Now we are ready to state a criterium for $\mathbb{Q}$-factoriality of $X = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}]$. For an algebraic group H, define its character group $\mathfrak{X}(H) := \operatorname{Hom}(H, \mathbb{C}^\times)$, where the Hom is taken in the category of algebraic groups. Note that $\mathfrak{X}(H) \xleftarrow{\sim} \mathfrak{X}(H/(H,H))$. Also, if U is a unipotent normal subgroup of H, then $\mathfrak{X}(H) \xleftarrow{\sim} \mathfrak{X}(H/U)$.

Proposition: Assume G is simply connected. Let $H \subset G$ be the stabilizer of a point in $\widetilde{\mathbb{O}}$. Set $X = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}]$. Then $\operatorname{Pic}(X) = \{0\}$ & $\operatorname{Pic}(X^{reg}) \simeq \mathfrak{X}(H)$. So X is $\mathbb{Q}$-factorial $\Leftrightarrow |\mathfrak{X}(H)| < \infty$.

Proof: • $\mathrm{Pic}(X)=\{0\}$: We have $\mathbb{C}^\times \curvearrowright X$ turning $\mathbb{C}[X]$ into a positively graded algebra. Every line bundle on X is $\mathbb{C}^\times$-linearizable, i.e. the corresponding module, say, M admits a grading. Let $\mathfrak{m}\subset \mathbb{C}[X]$ be the max ideal of $0\in X$. Note that $M/\mathfrak{m}_0 M\simeq \mathbb{C}$. Using the graded Nakayama lemma, we see that M is generated by one element. But M is projective, so we must have $M\simeq \mathbb{C}[X]$. This finishes the proof.

• $\mathrm{Pic}(X^{reg})\simeq \mathfrak{X}(H)$: Since $\mathrm{codim}_{X^{reg}} X^{reg}\setminus\widetilde{\mathbb{O}}\geqslant 2$, and X^{reg} is smooth, $\mathrm{Pic}(X^{reg})\simeq \mathrm{Pic}(G/H)$. We have the forgetful homomorphism $\mathrm{Pic}^G(G/H)\longrightarrow \mathrm{Pic}(G/H)$. It's surjective thx to Thm. It's also injective: the trivial line bundle only has one G-linearization, this follows b/c $\mathbb{C}[X]^\times/\mathbb{C}^\times$ is a finitely generated group, so G acts on it trivially. On the other hand, $\mathfrak{X}(G)=\{0\}$ so the section 1 of $\mathcal{O}_{G/H}$ is G-invariant. So, $\mathrm{Pic}^G(G/H)\xrightarrow{\sim}\mathrm{Pic}(G/H)$.

On the other hand, $\mathrm{Pic}^G(G/H)\xrightarrow{\sim}\mathfrak{X}(H)$ via $\mathcal{L}\mapsto$ fiber $\mathcal{L}_{1H}$. The inverse map is given by sending a 1-dimensional H-rep. V to the homogeneous line bundle $G\times^H V$.

So $\mathfrak{X}(H)\xrightarrow{\sim}\mathrm{Pic}^G(G/H)\xrightarrow{\sim}\mathrm{Pic}(G/H)$. This finishes the proof. □

Exercise: Let $\mathbb{O} \subset \mathfrak{g}$ be a nilpotent orbit & $X = \operatorname{Spec} \mathbb{C}[\mathbb{O}]$. Use results of lecture 6 (computing $Z_G(e,h,f)$) to show that:

- If $\mathfrak{g} \simeq \mathfrak{sl}_n$, then X is $\mathbb{Q}$-factorial $\Leftrightarrow$ $\mathbb{O}$ is principal.
- If $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$, then X is $\mathbb{Q}$-factorial.
- What happens for $\mathfrak{so}_{2n}$?

1.3) Main result.

The following is a consequence of a much more general result of [BCHM], see Proposition 2.1 in [L3].

Theorem: Let X be a conical symplectic singularity. For a partial Poisson resolution $Y \to X$ TFAE:

a) Y is maximal.

b) Y is $\mathbb{Q}$-factorial & terminal (a.k.a. Y is a $\mathbb{Q}$-factorial terminalization of X)

Moreover, Y satisfying these conditions always exists.

Remark: Y is often non-unique. We may discuss this in subsequent lectures.

Exercise: Prove (a)$\Leftrightarrow$(b) if Y is smooth (and hence is a symplectic resolution).

2) Cohomology vanishing.

In this short section we sketch of proof of the following.

Proposition: Let X be a conical symplectic singularity & Y is its partial Poisson resolution w. $\operatorname{codim}_Y Y^{sing} \geqslant 4$, where $Y^{sing} := Y \setminus Y^{reg}$. Then $\mathbb{C}[Y^{reg}] \xrightarrow{\sim} \mathbb{C}[X]$ & $H^i(Y^{reg}, \mathcal{O}_Y) = 0$ for $i=1,2$.

Proof: Let $\pi: Y \to X$, $\iota: Y^{reg} \hookrightarrow Y$ be the natural morphisms.

Step 1: Here we prove $R\pi_* \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X$. Let $\rho: Z \to Y$ be a resolution of singularities. Recall that Y is singular symplectic (Section 3.2 of Lec 11). So is X. Hence both X & Y have rational singularities (Sec 1.4 of Lec 9)

It follows that

$$R\rho_* \mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_Y \text{ \& } R(\pi\circ\rho)_* \mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_X.$$

Since $R(\pi\circ\rho)_* \simeq R\pi_* \circ R\rho_*$ we deduce $R\pi_* \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X$.

Step 2: Similarly, $R_*(\pi\circ\iota)\simeq R\pi_*\circ R\iota_*$. By the Hartogs Thm, $\iota_*\mathcal{O}_{Y^{reg}}\simeq\mathcal{O}_Y$. It remains to show that $R^i\iota_*\mathcal{O}_{Y^{reg}}=0$ for $i=1,2$. We have the endofunctor $\gamma: QCoh(Y)\longrightarrow QCoh(Y)$ that sends $\mathcal{F}\in Coh(Y)$ to its subsheaf of all local sections supported on Y. We have an exact sequence

$$0\to\gamma(\mathcal{F})\longrightarrow\mathcal{F}\longrightarrow\iota_*\iota^*\mathcal{F}$$

that is a SES on flabby sheaves, compare to Exer 2.3 in Ch. 3 of Hartschorne's book - we are doing a "local version" of that. So we get a distinguished triangle

$$R\gamma\,\mathcal{O}_Y\longrightarrow\mathcal{O}_Y\longrightarrow R\iota_*\mathcal{O}_{Y^{reg}}\xrightarrow{+1}$$

Since $\mathcal{O}_Y\xrightarrow{\sim}\iota_*\mathcal{O}_{Y^{reg}}$, we get $R^i\iota_*\mathcal{O}_{Y^{reg}}\xrightarrow{\sim}R^{i+1}\gamma\,\mathcal{O}_Y$.

Step 3: Since Y is singular symplectic, it's CM (see Sec 1 of Lec 9). It's known (e.g. Exer 3.4 in Ch. 3 of Hartschorne's book) that $R^{i+1}\gamma\,\mathcal{O}_Y=0$ if $i+1\leqslant\operatorname{codim}Y^{sing}=4$. It follows that $R^i\iota_*\mathcal{O}_{Y^{reg}}=0$ for $i=1,2$. Since X is affine, this implies $H^i(Y^{reg},\mathcal{O}_Y)=0$ for $i=1,2$. $\square$