Lecture 13

1) Classical Hamiltonian reduction.

2) Induced varieties.

Refs: [CdS], Ch. IX; [CM], Sec 7.

1.0) Discussion.

In the previous lecture we have seen that for every conical symplectic singularity X there is a maximal partial Poisson resolution, a.k.a. $\mathbb{Q}$-factorial terminalization, Y. Our next goal will be to construct such for $X=\mathbb{C}[\tilde{\mathbb{O}}]$, where $\tilde{\mathbb{O}}$ is a G-equivariant cover of a nilpotent orbit in $\mathfrak{g}^*$. The construction is a variant of "parabolic induction" that is based on "Hamiltonian reduction." This procedure that allows to construct more complicated Poisson varieties from easier ones was originally invented by Marsden and Weinstein in the C^∞-setting.

This is discussed in [CdS], Ch. IX.

1.1) Hamiltonian reduction of algebras.

Let H be an algebraic group, A be a Poisson algebra. Suppose that we have a rational Hamiltonian H-action on A. "Hamiltonian" means that we've fixed a comoment map: $\varphi: \mathfrak{h} \longrightarrow A$, H-equivariant linear map s.t. $\{\varphi(\xi), a\} = \xi_A a$, $\forall\, \xi \in \mathfrak{h}, a \in A$, where $\xi \mapsto \xi_A : \mathfrak{h} \to \operatorname{End}(A)$ is the differential of the representation of H in A (this is an algebraic analog of $\mathfrak{g} \longrightarrow \operatorname{Vect}(M)$ from Sec 2.1 of Lec 2).

Note that $A\varphi(\mathfrak{h}) \subset A$ is an H-stable ideal.

Definition/lemma: The Hamiltonian reduction $A /\!\!/\!\!/_0 H$ is the algebra $(A/A\varphi(\mathfrak{h}))^H$ w. Poisson bracket given by:

$$\{a + A\varphi(\mathfrak{h}), b + A\varphi(\mathfrak{h})\} := \{a, b\} + A\varphi(\mathfrak{h}).$$

Very important Exercise: Show that this bracket is well-defined (hint: the condition $a + A\varphi(\mathfrak{h}) \in (A/A\varphi(\mathfrak{h}))^H$ implies $\{\varphi(\xi), a\} = \xi_A a \in A\varphi(\mathfrak{h})$) and is a Poisson bracket.

Remarks: 1) Suppose that A is graded w. $\deg\{\cdot,\cdot\} = -d$,

H preserves the grading & $\operatorname{im}\varphi \in A_\alpha$. Then $A///_0 H$ is graded w. $\deg\{\cdot,\cdot\} = -\alpha$.

2) Pick $\chi \in (\mathfrak{h}^*)^H$. Then $\varphi_\chi : \mathfrak{h} \to A$ defined by $\varphi_\chi(\xi) = \varphi(\xi) - \langle \chi, \xi \rangle$ is also a comoment map (compare to Prob 1 in HW1). We write $A///_\chi H$ for $(A/A\varphi_\chi(\mathfrak{h}))^H$. So we get a family of Poisson algebras parameterized by $(\mathfrak{h}^*)^H$.

1.2) Hamiltonian reduction of schemes.

Now let X be a Poisson scheme equipped w. an algebraic action of an algebraic group H. In particular, we have an H-equivariant Lie algebra homomorphism $\xi \mapsto \xi_X : \mathfrak{h} \to \operatorname{Vect}(X)$, and so it makes sense to speak about Hamiltonian actions. Suppose $H \curvearrowright X$ is Hamiltonian & $\mu : X \to \mathfrak{h}^*$ be the moment map. If X is affine, then we are in the situation of the previous section.

Taking the quotient $A/A\varphi(\mathfrak{h})$ in the algebra setting corresponds to taking the closed subscheme $\mu^{-1}(0) \subset X$ (w. its natural scheme structure). Taking the invariants should correspond to taking the "quotient" $\mu^{-1}(0)/H$ (compare to Sec 2 of Lec 9). Taking quotients is tricky so we'll make a number of simplifying assumptions.

Assumption: There's a scheme Z s.t. we have a <u>principal H-bundle</u> $\mu^{-1}(0) \to Z$. This means that there's a surjective etale morphism $\tilde{Z} \to Z$ s.t. $\tilde{Z} \times_Z \mu^{-1}(0) \xrightarrow[H]{\sim} Z \times H$. We will mostly need the situation when $\mu^{-1}(0) \to Z$ is locally trivial in the Zariski topology (a stronger condition).

Under this assumption, we set $X/\!\!/\!\!/_0 H := Z$.

Exercise 1: Equip $X/\!\!/\!\!/_0 H$ with a Poisson structure. Moreover, show that if X is smooth & symplectic, then so is $X/\!\!/\!\!/_0 H$.

Examples: 0) The action of H on T^*H (say by right translations) is Hamiltonian w. $\varphi(\xi) = \xi_H$. These vector fields are left-invariant. So if we trivialize $T^*H \simeq H \times \mathfrak{h}^*$ using left-invariant vector fields, the moment map for this action is $(h,\alpha)\mapsto -\alpha$. Hence $\mu^{-1}(0)$ is the zero section H & $T^*H/\!\!/\!\!/_0 H = pt$.

1) Let $G\supset H$ be algebraic groups. Consider $H\curvearrowright T^*G$ by right translations. Then, under the trivialization similar to Ex 0, we get $\mu(g,\alpha)\mapsto -\alpha|_{\mathfrak{h}^*}$. So $\mu^{-1}(0)=G\times(\mathfrak{g}/\mathfrak{h})^*$. The action of H is diagonal: $h.(g,\alpha)=(gh^{-1}, h.\alpha)$, so $T^*G/\!\!/\!\!/_0 H = G\times^H(\mathfrak{g}/\mathfrak{h})^* =$ [compare to Rem 2 in Sec. 1 of Lec 10] $= T^*(G/H)$.

Exercise 2: Show that the resulting Poisson bracket on $T^*(G/H)$ is the standard one (hint: $G\to G/H$ is a principal H-bundle).

2) Induced varieties.

2.1) Parabolic subgroups (details: [OV], Exer. 20-27 in Sec. 4.2)

Let G be a connected reductive (e.g. s/simple) algebraic group. Recall that a subgroup $P\subset G$ is called parabolic if it contains a Borel subgroup; $\mathfrak{p}=Lie(P)\subset\mathfrak{g}$ is called a parabolic subalgebra. Here's a construction. Let $\Pi \subset \Delta^+ \subset \Delta \subset \mathfrak{h}^*$ be the systems of simple, positive, and all roots. Take a subset $\Pi_0 \subset \Pi$ and set $\Delta_0 := \mathrm{Span}_{\mathbb{Z}}(\Pi_0) \cap \Delta$. Form the subalgebra $\mathfrak{p}(\Pi_0) = \mathfrak{h} \oplus \bigoplus_{\beta \in \Delta_0 \cup \Delta_+} \mathfrak{g}_\beta$, it's parabolic. Every parabolic subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{p}(\Pi_0)$ for the unique Π_0. Set $\mathfrak{l}(\Pi_0) := \mathfrak{h} \oplus \bigoplus_{\beta \in \Delta_0} \mathfrak{g}_\beta$, $\mathfrak{n}(\Pi_0) := \bigoplus_{\beta \in \Delta^+ \setminus \Delta_0} \mathfrak{g}_\beta$, a subalgebra and an ideal of $\mathfrak{p}(\Pi_0)$. They correspond to algebraic subgroups $L(\Pi_0), U(\Pi_0)$ $\subset P(\Pi_0) \subset G$, where $L(\Pi_0)$ is reductive (it's a Levi subgroup) & $U(\Pi_0)$ is unipotent. We have $P(\Pi_0) = L(\Pi_0) \ltimes U(\Pi_0)$.

Example: For $G = GL_n$, the subgroups $P(\Pi_0), L(\Pi_0), U(\Pi_0)$ are subgroups of block upper triangular, block diagonal & block strictly unitriangular matrices (for some decomposition into blocks).

Exercise: Describe the subgroups $P(\Pi_0), L(\Pi_0), U(\Pi_0)$ for $G = SO_n$ & Sp_n.

Since every parabolic P is conjugate to some $P(\Pi_0)$, we have $P = L \ltimes U$ for Levi subgroup L & unipotent subgroup U.

2.2) Parabolic induction.

This construction allows to go from representation-theoretic objects for a Levi subgroup L (easier) to similar objects for the ambient group G (harder). It occurs in essentially every branch of Lie representation theory: the study of category $\mathcal{O}$, representation theory of reductive groups over finite/real/p-adic ... fields, representation theory of Weyl groups/Hecke algebras, etc.

There's a version of parabolic induction for nilpotent orbits due to Lusztig & Spaltenstein, see Sec 7, [CM]. In this course, we consider a certain upgrade.

2.3) Construction of induced varieties.

Fix a parabolic subgroup $P = L \ltimes U \subset G$. Let $\widetilde{\mathbb{O}}_L$ be an L-equivariant cover of a nilpotent orbit $\mathbb{O}_L \subset \mathfrak{l}^*$.

Remark: Recall, Sec. 1.3 in Lec 5, that $\mathbb{O}_L \subset (\mathfrak{l}/\mathfrak{z}(\mathfrak{l}))^*$. Moreover, the action of L on $\widetilde{\mathbb{O}}_L$ factors through the s/simple quotient $L/Z(L)^\circ$ (exercise). So we are still talking about equivariant covers of nilpotent orbits for s/simple groups.

Set $X_L := \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}_L]$. This is a Poisson variety w. moment map $\mu: X_L \to \mathfrak{l}^*$, a finite morphism w. image $\overline{\mathbb{O}}_L$.

Next, consider the quotient G/U. It comes w. commuting actions of G & L: $(g,\ell)hU = gh\ell^{-1}U$ (the L-action makes sense b/c L normalizes U.

The following lemma will be useful.

Lemma: The principal U-bundle $G \to G/U$ is locally trivial in Zariski topology. The same holds for the principal P-bundle $G \to G/P$.

Proof: Suppose $P = P(\Pi_0)$. Let P^- be the "opposite" parabolic subgroup - w. Lie algebra $\mathfrak{p}^- = \mathfrak{h} \oplus \bigoplus_{\beta \in -\Delta_+ \cup \Delta_0} \mathfrak{g}_\beta$. We get $P^- = L \ltimes U^-$ w. $\mathfrak{n}^- = \bigoplus_{\beta \in -\Delta_+ \setminus \Delta_0} \mathfrak{g}_\beta$ so that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}$.

Exercise 1: 1) The maps $U^- \to G/P$, $u \mapsto uP$ & $P^- \to G/U$, $p \mapsto pU$, are open embeddings.

2) $G \to G/P$ & $G \to G/U$ trivialize over the images of U^- & P^- respectively & over their translates by elements of G.

2) finishes the proof. $\square$

Consider the induced $G\times L$-action on $T^*(G/U)$. It's Hamiltonian (Sec 2.2 of Lec 2). The moment map can be described as follows. We can view $T^*(G/U)$ as the homogeneous vector bundle $G\times^U(\mathfrak{g}/\mathfrak{n})^*$. The moment map

$$(\mu_G,\mu_L): T^*(G/U) \longrightarrow \mathfrak{g}^*\times\mathfrak{l}^* \text{ is given by}$$

$$\mu_G([h,\alpha]) = h\alpha,\ \mu_L([h,\alpha]) = -\alpha|_{\mathfrak{l}^*},\ h\in G,\ \alpha\in(\mathfrak{g}/\mathfrak{n})^*.$$

where $\mathfrak{l}^*\hookrightarrow(\mathfrak{g}/\mathfrak{n})^*$ via $\mathfrak{g}/\mathfrak{n}\simeq\mathfrak{l}\oplus\mathfrak{n}^-$.

Now consider the diagonal action $G\times L \curvearrowright T^*(G/U)\times\tilde{X}_L$: $(g,\ell)([h,\alpha],x) = ([gh\ell^{-1}, \ell.\alpha], \ell.x)$. It's Hamiltonian w. moment maps $(\mu_G,\mu_L): ([h,\alpha],x)\mapsto(h.\alpha, -\alpha|_{\mathfrak{l}^*}+\mu(x))$

Definition: The induced variety $\mathrm{Ind}_P^G(\tilde{X}_L)$ is the Hamiltonian reduction $\mu_L^{-1}(0)/L$ $(\simeq G\times^P\{(\alpha,x)\mid \alpha|_{\mathfrak{l}^*}=\mu(x)\})$.

Example: Let $\tilde{X}_L = \{0\}$. Then $\mathrm{Ind}_P^G(\{0\}) = G \times^P (\mathfrak{g}/\mathfrak{p})^* = T^*(G/P)$ as Poisson varieties. Details are left as an exercise, for example you could try to show that

$$T^*(G/U) /\!\!/\!\!/_0 L \simeq [P = L \ltimes U] = T^*G /\!\!/\!\!/_0 P = T^*(G/P).$$