# Lecture 14.

1) Deformation of induced varieties.

2) Induced covers.

Refs: [CM], Sec 7.

## 1.0) Reminder & goal

$L \subset P = L \ltimes U \subset G$ , Levi & parabolic subgroups

$X_L = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}_L] \xrightarrow[\mu]{} \mathfrak{l}^*$

$\rightsquigarrow T^*(G/N) \times X_L \curvearrowleft G \times L,\ (g,\ell).([h,\alpha],x) = ([gh\ell^{-1}, \ell\alpha], \ell x)$

$(\mu_G, \mu_L): T^*(G/N) \times X_L \longrightarrow \mathfrak{g}^* \times \mathfrak{l}^*,\ ([h,\alpha],x) \mapsto (h\alpha, \mu(x) - \alpha|_{\mathfrak{l}^*})$

where $\alpha \mapsto \alpha|_{\mathfrak{l}^*}: (\mathfrak{g}/\mathfrak{n})^* \twoheadrightarrow \mathfrak{l}^*$, dual to inclusion $\mathfrak{l} \hookrightarrow \mathfrak{g}/\mathfrak{n}$.

We set $Y := \operatorname{Ind}_P^G(X_L) = \mu_L^{-1}(0)/L$

$= G \times^P \{(\alpha, x) \in (\mathfrak{g}/\mathfrak{n})^* \times X_L \text{ s.t. } \alpha|_{\mathfrak{l}^*} = \mu(x)\}.$

$G \curvearrowright Y$: $g.([h,(\alpha,x)]) = [gh,(\alpha,x)]$.

We'll see that $Y$ has the unique open $G$-orbit, which is a $G$-equivariant cover of a nilpotent orbit.

## 1.1) Deformations.

Exercise 1 (on understanding the Hamiltonian reduction)

$\mu_G: Y \longrightarrow \mathfrak{g}^*, [h,(\alpha,x)] \mapsto h\alpha$ is a moment map.

Next, we need a deformation of $Y$. Pick $\chi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ and define $Y_\chi := \mu_L^{-1}(-\chi)/L$

$= G\times^P\{(\alpha,x)\in(\mathfrak{g}/\mathfrak{n})^*\times X_L \text{ s.t. } \alpha|_{\mathfrak{l}^*} = \mu(x)+\chi\}$

This is also a Poisson variety & it has Hamiltonian $G$-action w. $\mu_G([h,(\alpha,x)]) = h\alpha$.

Even better, we can consider the "universal" Hamiltonian reduction. Set $\mathfrak{z} := (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$, this embeds into $\mathfrak{l}^*$. Set

$Y_\mathfrak{z} := G\times^P\{(\alpha,x)\in(\mathfrak{g}/\mathfrak{n})^*\times X_L \mid \alpha|_{\mathfrak{l}^*} - \mu(x)\in\mathfrak{z}\}$

The map $[h,(\alpha,x)] \mapsto \alpha|_{\mathfrak{l}^*} - \mu(x)$ realizes $Y_\mathfrak{z}$ as a scheme over $\mathfrak{z}$ w. fiber $Y_\chi$ over $\chi\in\mathfrak{z}$.

Example: Let $L=T$, $P=B$. We can only take $X_L=\{0\}$. Then $Y = T^*(G/B)$ $\mathfrak{z}=\mathfrak{h}^*$ & $Y_\mathfrak{z} = G\times^B(\mathfrak{g}/\mathfrak{n})^*$ w. $Y_\mathfrak{z} \twoheadrightarrow \mathfrak{z}$: $[h,\alpha]\to\alpha|_\mathfrak{h}$. The map $Y_\mathfrak{z}\to\mathfrak{g}^*$: $[h,\alpha]\to h\alpha$ is called the Grothendieck simultaneous resolution, it deforms the Springer resolution $Y \twoheadrightarrow \mathcal{N}$ from Sec 1 of Lec 10.

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Exercise 2: $Y_\mathfrak{z}$ is a Poisson scheme over $\mathfrak{z}$ meaning that $Y_\mathfrak{z}$ is Poisson & pullbacks of functions from $\mathfrak{z}$ are central.
Hint: do universal reduction construction for algebras first.

Remark: $Y_\mathfrak{z} \to \mathfrak{z}$ is flat: $Y_\mathfrak{z}$ is CM, $\mathfrak{z}$ is smooth & all fibers have the same dimension: $\dim G/P + (\dim X_L + \dim (\mathfrak{g}/\mathfrak{p})^*)$: see [E], Thm 18.16. For the dimension formula note that we have SES $0 \to (\mathfrak{g}/\mathfrak{p})^* \to (\mathfrak{g}/\mathfrak{h})^* \to \mathfrak{l}^* \to 0$ so in the bracket we have the dimension of fiber of $Y_\chi \to G/P$.

## 2) Induced covers.

Our goal in this section is to prove the following result

Theorem: Let $\chi \in \mathfrak{z} (\xrightarrow{\sim} \mathfrak{z}(\mathfrak{l})$ under $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*)$

1) Let $\mu_G : Y_\chi \to \mathfrak{g}^*$ be the moment map. Then $\operatorname{im} \mu_G = \overline{\mathbb{O}}$ for adjoint orbit $\mathbb{O} \subset \mathfrak{g}$ s.t. for $x \in \mathbb{O}$ we have $Gx_s = G\chi$.

2) $\exists!$ open $G$-orbit $\tilde{\mathbb{O}} \subset Y_\chi$ & $\mu_G : \tilde{\mathbb{O}} \twoheadrightarrow \mathbb{O}$ is $G$-equiv't cover

3) Set $X = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$. Then $\mu_G : Y_\chi \to \overline{\mathbb{O}}$ factors as $Y_\chi \xrightarrow{\pi} X \longrightarrow \overline{\mathbb{O}}$ (Stein factorization), where $\pi$ is a partial Poisson resolution.

Ex: $L=T$, $P=B$, $\chi=0$. Then $\operatorname{im}\mu=\mathcal{N}$, $\tilde{\mathbb{O}}=\mathbb{O}_{pr}$, Sec 1 in Lec 10.

## 2.1) Proof of 1)

Exercise: The morphism $\mu_G: Y_\mathfrak{z}\longrightarrow\mathfrak{g}^*$, $[h,(\alpha,x)]\mapsto h\alpha$, is projective.
Hint: compare to the proof of 1) of Thm in Sec 2 of Lec 8

Proof:

$Y_\chi$ is irreducible. Note that under $\mathfrak{g}\simeq\mathfrak{g}^*$ we have $(\mathfrak{g}/\mathfrak{p})^*\simeq\mathfrak{n}$ & $\mathfrak{l}^*\simeq\mathfrak{l}$ so $\alpha\in\chi+\overline{\mathbb{O}}_L+\mathfrak{n}$. We have a grading on $\mathfrak{g}$ w. $\mathfrak{l}$ in deg 0, & $\mathfrak{n}$ in positive degrees. Indeed, we can assume $\mathfrak{p}=\mathfrak{p}(\Pi_0)$ (Sec 2.1 of Lec 13). Take a coweight $x\in\mathfrak{h}$ s.t. $x|_{\Pi_0}=0$ & $x|_{\Pi\setminus\Pi_0}>0$. Consider the corresponding homomorphism $\gamma:\mathbb{C}^\times\longrightarrow G$ (so that $d_1\gamma(1)=x$). Let $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be the grading by eigenvalues of $x$ so that $\mathfrak{l}=\mathfrak{g}_0$, $\mathfrak{n}=\bigoplus_{i>0}\mathfrak{g}_i$. Write $\alpha=\sum_{i\geqslant 0}\alpha_i$. Then $\alpha_0=\lim_{z\to 0}\gamma(z)\in\overline{G\alpha}$. We have that $\alpha_0=\chi+(\alpha_0-\chi)$ is the Jordan decomposition ($\alpha_0-\chi\in\overline{\mathbb{O}}_L$). So $\overline{G\alpha}\cap\pi_G^{-1}(\pi_G(\chi))\neq\emptyset\Rightarrow$ $\pi_G(\alpha)=\pi_G(\chi)\Rightarrow G\alpha_s=G\chi$. $\square$

## 2.2) Proof of 2).

Thx to 1), the proof reduces to checking

$$\dim \mu_G(Y_\chi) = \dim Y_\chi \qquad (1)$$

We write $\mathbb{O}_\chi$ for the dense orbit in $\mu_G(Y_\chi)$, so $\mu_G(Y_\chi)=\overline{\mathbb{O}}_\chi$.

Case 1: $\chi$ is generic: $\mathfrak{z}_\mathfrak{g}(\chi)=\mathfrak{l}$. We claim that $\mu_G: Y_\chi \to \overline{\mathbb{O}}_\chi$ is finite, which implies (1). Let $\eta: (\mathfrak{g}/\mathfrak{n})^* \twoheadrightarrow \mathfrak{l}^*$ be the natural projection so that the condition on $\alpha$ is $\alpha\in \eta^{-1}(\overline{\mathbb{O}}_L+\chi)$. The morphism $\mu_G$ factors through a finite morphism $Y_\chi \twoheadrightarrow G\times^P \eta^{-1}(\overline{\mathbb{O}}_L+\chi)$, $[h,(\alpha,x)]\mapsto [h,\alpha]$. It remains to show that $G\times^P\eta^{-1}(\overline{\mathbb{O}}_L+\chi) \to \overline{\mathbb{O}}_\chi$ is isomorphism. Consider the action map $P\times(\overline{\mathbb{O}}_L+\chi) \to \eta^{-1}(\overline{\mathbb{O}}_L+\chi)$. It factors through

$$P\times^L(\overline{\mathbb{O}}_L+\chi)(\simeq N\times(\overline{\mathbb{O}}_L+\chi)) \to \eta^{-1}(\overline{\mathbb{O}}_L+\chi)(=\chi+\overline{\mathbb{O}}_L+\mathfrak{n}) \qquad (2)$$

Exercise: • Use $[\mathfrak{n},\chi]=\mathfrak{n}$ to show (2) is an isomorphism

• Deduce that $G\times^P\eta^{-1}(\overline{\mathbb{O}}_L+\chi) = G\times^L(\chi+\overline{\mathbb{O}}_L) \to \overline{\mathbb{O}}_\chi$ is an isomorphism (hint: compare to the proof of Proposition in Sec 1.3 of Lec 5)

Case 2: $\chi=0$. Consider $Y_{\mathbb{C}\chi'} := \mathbb{C}\chi' \times_{\mathfrak{z}} Y_{\mathfrak{z}}$ for generic $\chi'$. The map

$$\mu_G : Y_{\mathbb{C}\chi'} \longrightarrow \mathfrak{g}^*$$

is projective by Exercise in Sec 2.1, it's image is $\bigcup_{z\in\mathbb{C}} \overline{\mathbb{O}}_{z\chi'}$.
Note that $\pi_G(\overline{\mathbb{O}}_{z\chi'}) = \pi_G(z\chi')$. So $\operatorname{im}(\pi_G \circ \mu_G) \subset \mathfrak{g}/\!/G$ is a curve, denote it by $C$. The variety $\operatorname{im}\mu_G$ is irreducible b/c $Y_{\mathbb{C}\chi'}$ is, so $\dim \operatorname{im}\mu_G = \dim \overline{\mathbb{O}}_{\chi'} + 1 = [\text{Case 1}] = \dim Y_{\chi'} + 1 = \dim Y + 1$. On the other hand, $\overline{\mathbb{O}}$ is the $0$-fiber of $\pi_G : \operatorname{im}\mu_G \longrightarrow C$ & its dimension $\geqslant$ that of general fiber $= \dim \overline{\mathbb{O}}_{z\chi'} = \dim Y_{z\chi'} = \dim Y$. So

$$\dim \overline{\mathbb{O}} \geqslant \dim Y$$

Since $\overline{\mathbb{O}} = \mu_G(Y)$ we have $\dim \overline{\mathbb{O}} = \dim Y$ finishing the proof in this case.

Case 3: general $\chi$.

Exercise: Let $G$ be an algebraic group acting on a variety $Z$. Let $d = \max_{z\in Z} \dim G z$. Then $Z^\circ := \{z \in Z \mid \dim Gz = d\}$ is open.

Apply this to $G \curvearrowright Y_{\mathfrak{z}}$. Note that $\mathbb{C}^\times \curvearrowright Y_{\mathfrak{z}}$. For this recall that $\mathbb{C}^\times \curvearrowright X_L$ w. $\mu(t.x) = t^2\mu(x)$ (see Sec. 2 of Lec 7)
Now set $t.[h,(\alpha,x)] = [h, (t^2\alpha, t.x)]$. This action commutes w.

$G$. So $Y_z^\circ$ is $\mathbb{C}^\times$-stable. By Case 1, $d = \dim Y$. Now consider $Y_{\mathbb{C}X}$. Since $Y \subset Y_{\mathbb{C}X} \subset Y_z$, the maximal dimension of the $G$-orbit in $Y_{\mathbb{C}X}$ is $\geqslant$ that in $Y$ & $\leqslant$ that in $Y_z$. Both are equal to $d$, so the maximal dim of a $G$-orbit in $Y_{\mathbb{C}X}$ is $d$ as well. But $Y_{\mathbb{C}X} \to \mathbb{C}X$ is $\mathbb{C}^\times$-equivariant. One of nonzero fibers contains a dimension $d$ $G$-orbit, by Exercise, hence all of them must. So every fiber of $Y_z \to z$ contains an orbit of dimension $d = \dim Y$, which finishes the proof. $\square$

## 2.3) Proof of 3).

We write $\widetilde{\mathbb{O}}_X$ for the open orbit in $Y_X$. The open inclusion

$$\widetilde{\mathbb{O}}_X \hookrightarrow \mathbb{O}_X \times_{M_G(Y_X)} Y_X \qquad (3)$$

is an isomorphism (exercise). In particular, a generic fiber of $Y_X \to \overline{\mathbb{O}}_X$ is finite. Note that $Y_X$ is normal b/c $X_L$ is (exercise). The Stein factorization (Hartshorne, Ch. 3, Sec. 11) tells us that $Y_X \to \overline{\mathbb{O}}_X$ factors as $Y_X \longrightarrow \widetilde{X} \longrightarrow \overline{\mathbb{O}}_{X'}$, where the fibers of $Y_X \to \widetilde{X}$ are connected, while $\widetilde{X} \to \overline{\mathbb{O}}_X$ is finite. The variety $\widetilde{X}$ is normal. Since (3) is an iso, $\widetilde{\mathbb{O}}_X \hookrightarrow \widetilde{X}$. The construction of $\widetilde{X}$ is canonical so $G \curvearrowright \widetilde{X}$ (to check the action is algebraic requires a bit of work) & $Y_X \longrightarrow \tilde{X} \longrightarrow \overline{\mathbb{O}}_X$ are $G$-equivariant. So $\tilde{\mathbb{O}}_X$ is an open orbit in $\tilde{X}$ & $\operatorname{codim}_{\tilde{X}} \tilde{X}\setminus\tilde{\mathbb{O}}_X \geqslant 2$ b/c codim $\operatorname{codim}_{\overline{\mathbb{O}}_X} \overline{\mathbb{O}}_X \setminus \mathbb{O}_X \geqslant 2$. Since $\tilde{X}$ is normal, $\tilde{X} = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}_X]$, finishing the proof.

## 2.4) Induced covers.

**Definition**: By the induced cover from $(L, \tilde{\mathbb{O}}_L, \chi)$ we mean $\tilde{\mathbb{O}}_X$. The notation is $\operatorname{Ind}_L^G(\tilde{\mathbb{O}}_L, \chi)$ (if $\chi = 0$, we drop it from notation).

More on induced covers in the next lecture.