Lecture 15

1) Properties of induced covers

2) Filtered deformations of $\mathbb{C}[\tilde{\mathbb{O}}]$

Refs: [CM], Sec. 7.1; [L1].

1.0) Reminder & goals.

We choose Levi & parabolic subgroups $L\subset P\subset G$, as well as an L-equivariant cover $\tilde{\mathbb{O}}_L$ of a nilpotent orbit $\mathbb{O}_L\subset \mathfrak{l}^*$.

Set $X_L:=\operatorname{Spec}\mathbb{C}[\tilde{\mathbb{O}}_L]$ and let $\mu: X_L\twoheadrightarrow \overline{\mathbb{O}}_L\hookrightarrow \mathfrak{l}^*$ be the natural map, it's a moment map. Set $\mathfrak{z}:=(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$.

Consider the variety $Y_{\mathfrak{z}}:=G\times^P\{(\alpha,x)\in(\mathfrak{g}/\mathfrak{n})^*\times X_L \mid \alpha|_{\mathfrak{l}}-\mu(x)\in\mathfrak{z}\}$. Here $P=L\ltimes U$, $\mathfrak{n}=\operatorname{Lie}(U)$. A point in $Y_{\mathfrak{z}}$ is a P-orbit of (h,α,x), $h\in G$, under the action $p.(h,\alpha,x)=(hp^{-1},p.\alpha,\pi(p)x)$. Here $\pi: P\twoheadrightarrow L$. Note that $(\mathfrak{g}/\mathfrak{n})^*\to\mathfrak{l}^*$, $\alpha\mapsto\alpha|_{\mathfrak{l}}$, is P-equivariant, so $\{(\alpha,x)\mid \alpha|_{\mathfrak{l}}-\mu(x)\in\mathfrak{z}\}\subset(\mathfrak{g}/\mathfrak{n})^*\times X_L$ is P-stable, so the action is well-defined. The variety $Y_{\mathfrak{z}}$ carries the following structures:

- A morphism $Y_{\mathfrak{z}}\to\mathfrak{z}$, $[h,(\alpha,x)]\mapsto\alpha|_{\mathfrak{l}}-\mu(x)$. Let Y_χ be the fiber of $\chi\in\mathfrak{z}$.

• A Hamiltonian G-action $G \curvearrowright Y_\mathfrak{z}$: $g.[h,(\alpha,x)] = [gh,(\alpha,x)]$ w. moment map $\mu: Y_\mathfrak{z} \to \mathfrak{g}^*$: $\mu([h,(\alpha,x)]) = h\alpha$.

We have seen (Thm in Sec 2 of Lec 15) that each Y_χ has an open G-orbit, $\tilde{\mathbb{O}}_\chi$, which is a G-equivariant cover of an orbit in $\mathfrak{g}^*$ whose semisimple part is in $G\chi$.

Definition: By the induced cover from $(L, \tilde{\mathbb{O}}_L, \chi)$ we mean $\tilde{\mathbb{O}}_\chi$. The notation is $\mathrm{Ind}_L^G(\tilde{\mathbb{O}}_L, \chi)$ (if $\chi=0$, we drop it from notation).

Here's how we apply induced varieties/covers to study various questions about covers:

• We'll see that for each equivariant cover $\tilde{\mathbb{O}}'$ of a coadjoint orbit in $\mathfrak{g}^*$ $\exists$ equivariant cover $\tilde{\mathbb{O}}$ of a nilpotent orbit s.t. $\mathbb{C}[\tilde{\mathbb{O}}']$ is a filtered Poisson deformation of $\mathbb{C}[\tilde{\mathbb{O}}]$ This will be done in this lecture

• We'll see that we can construct a $\mathbb{Q}$-factorial terminalization of $\mathrm{Spec}\,\mathbb{C}[\tilde{\mathbb{O}}]$ (where $\tilde{\mathbb{O}}$ is an equivariant cover of a nilpotent orbit) as an induced variety. This will be done in the next couple of lectures

• Then we'll use induction to construct quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$.

1.1) Independence of P.

$\mathrm{Ind}_L^G(\tilde{\mathbb{O}}_L, \mathcal{X})$ is independent of the choice of P, Lemma 4.1 in [L1]. That the underlying nilpotent orbit in $\mathfrak{g}$ is independent of P is proved in [CM], Section 7.1.

Example: Let $G = SL_n$. Up to conjugation, parabolic subgroups are subgroups of block upper triangular matrices and so correspond to compositions $n = n_1 + \ldots + n_k$. Let P be the corresponding parabolic & τ be the corresponding partition: n_i's in the decreasing order. Take $\mathbb{O}_L = \{0\}$, so that $Y = T^*(G/P)$.

Exercise 1: Show that:

1) $\mathrm{im}\,\mu = \overline{\mathbb{O}}_{\tau^t}$, the closure of orbit w. Jordan type τ^t.

Hint: use that $\dim \mathrm{im}\,\mu = \dim T^*(G/P)$ & check $\dim T^*(G/P) = \dim \overline{\mathbb{O}}_{\tau^t}$, Then for a Jordan matrix $J \in \mathbb{O}_{\tau^t}$, find subspaces $\mathbb{C}^n \supset V_1 \supset V_2 \supset \ldots \supset V_k = \{0\}$ s.t. $\mathrm{codim}\, V_i = n_1 + \ldots + n_i$ & $JV_i \subset V_{i+1}$; deduce $J \in \mathrm{im}\,\mu$.

2) $\widetilde{\mathbb{O}} = \mathbb{O}_{\tau^t}$. Hint: use that the centralizers of nilpotent elements in PGL_n are connected.

1.2) Transitivity of induction.

Take Levi subgroups $L \subset M \subset G$. So if $\xi \in \mathfrak{g}$ is s/simple el't s.t. $L = Z_G(\xi)$, then $\xi \in \mathfrak{l} \subset \mathfrak{m} \Rightarrow L = Z_M(\xi)$. It follows that $\mathfrak{z}(\mathfrak{m}) \subset \mathfrak{z}(\mathfrak{l})$. Using the Killing form of $\mathfrak{g}$ to identify $\mathfrak{l}^* \simeq \mathfrak{l}$, $\mathfrak{m}^* \simeq \mathfrak{m}$, we get $(\mathfrak{m}/[\mathfrak{m},\mathfrak{m}])^* \hookrightarrow (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$.

Lemma: The induction is transitive, e.g. for $\lambda \in (\mathfrak{m}/[\mathfrak{m},\mathfrak{m}])^* (\subset ([\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*)$ we have

$$\operatorname{Ind}_L^G(\widetilde{\mathbb{O}}_L, \lambda) \xrightarrow{\sim} \operatorname{Ind}_M^G(\operatorname{Ind}_L^M(\widetilde{\mathbb{O}}_L), \lambda)$$

Proof:

Pick parabolic subgroups $Q \subset G$ w. Levi M so that $Q = M \ltimes V$ & $P' \subset M$ w. Levi L so that $P' = L \ltimes U'$. Then $P = P' \ltimes U$ is a parabolic in G w. Levi L. For example, let

$$M = \left\{\begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}\right\}, \quad L = \left\{\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}\right\} \text{ in } SL_4. \text{ Then we take}$$

$$Q = \left\{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}\right\}, \quad P' = \left\{\begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}\right\}, \quad P = \left\{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}\right\}$$

Let $Y' = \mathrm{Ind}_{P'}^{M}(X_L) = M \times^{P'} \{(\beta, x) \in (\mathfrak{m}/\mathfrak{n}')^* \times X_L \mid \beta|_{\mathfrak{l}} = \mu(x)\}$

Then $\mathrm{Ind}_M^G(\mathrm{Ind}_L^M(\tilde{\mathbb{O}}_L), X)$ is the open G-orbit in

$$(T^*(G/V) \times Y')///_\lambda M \qquad (1)$$

Indeed, $\tilde{\mathbb{O}}_M := \mathrm{Ind}_L^M(\tilde{\mathbb{O}}_L)$ is the open M-orbit in Y'. Then $(T^*(G/V)\times\tilde{\mathbb{O}}_M)///_\lambda M$ is an open G-stable subvariety of $\mathrm{Ind}_Q^G(X_M, \lambda)$ & (1) so $\mathrm{Ind}_M^G(\tilde{\mathbb{O}}_M, \lambda)$ is the open G-orbit in (1).

A point in $\mu_M^{-1}(-\lambda)/M$ is: $[g, \gamma, h, \beta, x]$ w. $g\in G$, $\gamma \in (\mathfrak{g}/\mathfrak{v})^*$, $h \in M$, $\beta \in (\mathfrak{m}/\mathfrak{n}')^*$, $x \in X_L$ s.t. $\gamma|_{\mathfrak{m}} = \underbrace{h\beta}_{\text{moment map for } M \curvearrowright Y'} + \lambda$, $\beta|_{\mathfrak{l}} = \mu(x)$, where we identify $[g,\gamma,h,\beta,x]$ w. $[gv^{-1}m^{-1}, m\gamma, mhu'^{-1}\ell^{-1}, \ell\beta, \ell x]$ w. $v \in V$, $m\in M$, $u' \in U'$, $\ell \in L$. We can assume $h=1$ and then the conjugation is by P-action $[g,\gamma,\beta,x] = [gu^{-1}\ell^{-1}, \ell\gamma, \ell\beta, \ell x]$, $u \in U$, $\ell\in L$, while the condition becomes $\beta = \gamma|_{\mathfrak{m}} - \lambda$ & $\gamma|_{\mathfrak{l}} = \mu(x) + \lambda$; $\beta, \lambda \in (\mathfrak{m}/\mathfrak{n}')^* \Rightarrow \gamma \in (\mathfrak{g}/\mathfrak{n})^* (\subset (\mathfrak{g}/\mathfrak{v})^*)$. An isomorphism of (1) w. Y_λ is then given by $[g,\gamma,\beta,x] \mapsto [g,\gamma,x]$. It's G-equivariant (G acts on the 1st factor) & intertwines the moment maps (both are given by $g\gamma$), & so is an isomorphism of covers. $\square$

1.3) Non-nilpotent covers are induced.

It turns out that every cover of a non-nilpotent orbit is induced from a cover of a nilpotent one in a Levi.

Let $\mathbb{O}'$ be a non-nilpotent orbit in $\mathfrak{g}^*\simeq\mathfrak{g}$ and $\widetilde{\mathbb{O}}'\xrightarrow{\pi}\mathbb{O}'$ be a G-equivariant cover. Take $L:=Z_G(\xi_s)$ for $\xi\in\mathbb{O}'$ & let $\mathbb{O}_L$ to be the nilpotent orbit in $\mathfrak{l}$ w. $\mathbb{O}'=G(\xi_s+\mathbb{O}_L)$, see Sec 1.3 in Lec 5. Set $\widetilde{\mathbb{O}}_L:=\pi^{-1}(\xi_s+\mathbb{O}_L)$, this is an L-equivariant cover of $\mathbb{O}_L$ via $x\mapsto\mu(x):=\pi(x)-\xi_s$.

Exercise: • We have a natural iso $G\times^L\widetilde{\mathbb{O}}_L\longrightarrow\widetilde{\mathbb{O}}'$ (note that $G\times^L\mathbb{O}_L\longrightarrow\mathbb{O}'$, this follows from the argument in Sec 1.3, Lec 5).

• We have a P-equivariant isomorphism

$$\{(\mathfrak{g}/\mathfrak{n})^*\times\widetilde{\mathbb{O}}_L \mid \alpha|_{\mathfrak{l}}=\mu(x)+\xi_s\}\xrightarrow{\sim}P\times^L\widetilde{\mathbb{O}}_L$$

(compare to Case 1 in Sec 2.2 of Lec 14) yielding

$$\mathrm{Ind}_L^G(\widetilde{\mathbb{O}}_L,\xi_s)\xrightarrow{\sim}\widetilde{\mathbb{O}}'.$$

2) Filtered deformations of $\mathbb{C}[\widetilde{\mathbb{O}}]$

Fix $L,\widetilde{\mathbb{O}}_L$ w. $\mathrm{Ind}_L^G(\widetilde{\mathbb{O}}_L)=\widetilde{\mathbb{O}}$. Pick a parabolic subgroup P w. $P=L\ltimes U$. Pick $\chi\in\mathfrak{z}$ & set $Y_{\mathbb{C}\chi}:=\mathbb{C}\chi\times_{\mathfrak{z}}Y_{\mathfrak{z}}$. Recall that $\mathbb{C}^\times\curvearrowright Y_{\mathfrak{z}}$ (Sec. 2.3 of Lec 14). It restricts to $Y_{\mathbb{C}\chi}$:

$t.([h,(\alpha,x)])=[h,(t^2\alpha,t.x)]$

Note that Y is $\mathbb{C}^\times$-stable. Since the $\mathbb{C}^\times$- & G-actions on Y commute, the unique open G-orbit $\tilde{\mathbb{O}} \subset Y$ is $\mathbb{C}^\times$-stable.

Exercise 1: Show that $\mu_G: Y_z \to \mathfrak{g}^*$ is $\mathbb{C}^\times$-equivariant, where $\mathbb{C}^\times \curvearrowright \mathfrak{g}^*$ via $t.\alpha = t^2\alpha$. Deduce that the $\mathbb{C}^\times$-action on $\tilde{\mathbb{O}}$ introduced above coincides with the one from Sec 2.2 of Lec 7.

In particular, $\mathbb{C}[Y] = \mathbb{C}[\tilde{\mathbb{O}}]$ is positively graded.

Let z be the coordinate on $\mathbb{C}\chi$ w. $z(\chi)=1$. We can view z as a deg 2 homogeneous element in $\mathbb{C}[Y_{\mathbb{C}\chi}]$ via pullback. Observe that $\mathcal{O}_{Y_{\mathbb{C}\chi}}$ has no zero divisors b/c $Y_{\mathbb{C}\chi}$ is irreducible.

Proposition: 1) We have $\mathbb{C}[Y_{\mathbb{C}\chi}]/(z) \xrightarrow{\sim} \mathbb{C}[\tilde{\mathbb{O}}]$ as isomorphism of graded Poisson algebras.

2) We have $\mathbb{C}[Y_{\mathbb{C}\chi}]/(z-1) \xrightarrow{\sim} \mathbb{C}[\tilde{\mathbb{O}}_\chi]$, Poisson algebra iso.

Proof: 1) We have $\mathbb{C}[Y] \xrightarrow{\sim} \mathbb{C}[\tilde{\mathbb{O}}]$, graded Poisson algebra iso. We can view $\mathcal{O}_Y$ as a quotient of $\mathcal{O}_{Y_{\mathbb{C}\chi}}$, more precisely we have a SES: $0 \to \mathcal{O}_{Y_{\mathbb{C}\chi}} \xrightarrow{z} \mathcal{O}_{Y_{\mathbb{C}\chi}} \to \mathcal{O}_Y \to 0 \rightsquigarrow$ long exact sequence

$$0 \to \mathbb{C}[Y_{\mathbb{C}\chi}] \xrightarrow{z} \mathbb{C}[Y_{\mathbb{C}\chi}] \to \mathbb{C}[Y] \to H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}}) \xrightarrow{z} H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}}) \to H^1(\mathcal{O}_Y)$$

It's enough to show that $H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}})=0$. By Prop'n in Sec 2 of Lec 12, $H^1(\mathcal{O}_Y)=0$. So

$$z H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}}) = H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}}). \quad (1)$$

On the other hand, $H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}})$ is a fin. generated module over $\mathbb{C}[\mathfrak{g}^*]$. This is b/c $Y_{\mathbb{C}\chi}$ is projective over the affine $\mathfrak{g}^*$ (Exercise in Sec 2.1 of Lec 14), then we can use Theorem 5.2 in Sec 3 of Hartschorne). The $\mathbb{C}[\mathfrak{g}^*]$-action of $H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}})$ factors through $\mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[Y_{\mathbb{C}\chi}]$, so $H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}})$ is finitely generated over $\mathbb{C}[Y_{\mathbb{C}\chi}]$ as well.

Moreover, $H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}})$ is a graded module:
the action $\mathbb{C}^\times \curvearrowright Y_{\mathbb{C}\chi}$ gives rise to $\mathbb{C}^\times \curvearrowright H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}})$. It's rational: $Y_{\mathbb{C}\chi} \to G/P$ is a $\mathbb{C}^\times$-invariant affine morphism, so $Y_{\mathbb{C}\chi}$ can be covered by $\mathbb{C}^\times$-stable open affines $\rightsquigarrow$ $\mathbb{C}^\times \curvearrowright$ terms of Čech complex rationally.

The algebra $\mathbb{C}[Y_{\mathbb{C}\chi}]$ is $\mathbb{Z}_{\geq 0}$-graded: $\deg z = 2$ & $\mathbb{C}[Y_{\mathbb{C}\chi}]/(z) \hookrightarrow \mathbb{C}[Y]$, which is $\mathbb{Z}_{\geq 0}$-graded. So the grading on $H^1(\mathcal{O}_{Y_{\mathbb{C}\chi}})$ is bounded from below. And $(1) \Rightarrow H^1(\mathcal{O}_{\mathbb{C}\chi})=0$. This proves (1).

2): exercise (hint: write a similar exact sequence) $\square$

Thx to (2), the algebra $\mathbb{C}[Y_\chi]=\mathbb{C}[\tilde{\mathbb{O}}_\chi]$ inherits a filtration from the grading on $\mathbb{C}[Y_{\mathbb{C}\chi}]$. We leave it as an exercise to produce a graded Poisson isomorphism $\mathbb{C}[Y]\xrightarrow{\sim}\operatorname{gr}\mathbb{C}[Y_\chi]$ (use 1) of Prop'n - and compare to Exercise in Sec 1.2 of Lec 3). So $\mathbb{C}[Y_\chi]=\mathbb{C}[\tilde{\mathbb{O}}_\chi]$ becomes a filtered Poisson deformation of $\mathbb{C}[\tilde{\mathbb{O}}]$. Thx to Sec 1.3. we see that $\forall\ \tilde{\mathbb{O}}'$ as in there, $\mathbb{C}[\tilde{\mathbb{O}}']$ has a filtration making it a filtered Poisson deformation for suitable $\mathbb{C}[\tilde{\mathbb{O}}]$.