Lecture 16.

1) $\mathbb{Q}$-factorial terminalizations from induction, I.

Refs: [N2a], [N2b].

1.1) Main result

Let $\widetilde{\mathbb{O}}$ be a G-equivariant cover of a nilpotent orbit $\mathbb{O} \subset \mathfrak{g}^*$. Let $L \subset G$ be a <u>minimal</u> Levi subgroup s.t. $\exists$ L-equiv. cover s.t. $\widetilde{\mathbb{O}} = \operatorname{Ind}_L^G(\widetilde{\mathbb{O}}_L)$. Include L into a parabolic $P = L \ltimes U$.

Thm 1: The variety $Y = \operatorname{Ind}_P^G(X_L)$ w. $X_L = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}_L]$ is a $\mathbb{Q}$-factorial terminalization of $X = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}]$.

Examples: 1) Let $\widetilde{\mathbb{O}} = \mathbb{O}_{pr}$. In this case, can take $L = T$, $\widetilde{\mathbb{O}}_L = \{0\}$; L is certainly minimal. We have that $Y = T^*(G/B)$ is a symplectic resolution of $X = \mathcal{N}$.

2) $\mathfrak{g} = \mathfrak{sl}_n$, $X = \overline{\mathbb{O}}_{\tau^t}$ (see Example in Sec 1.1 of Lec 15). Then L is the subgroup of block diagonal matrices, where

sizes are the parts of τ. We have $\tilde{\mathbb{O}}_L = \{0\}$ and recover the fact that $Y = T^*(G/P)$ is a symplectic resolution of X.

1.2) Reduction to birationally rigid covers.

We only need to prove that Y is $\mathbb{Q}$-factorial terminal. Here's a reduction.

Proposition: Suppose X_L is $\mathbb{Q}$-factorial & terminal. Then so is Y.

Proof: We'll need to understand Y locally.

Let U^- be the opposite to P unipotent subgroup: if $P = P(\Pi_0)$ for $\Pi_0 \subset \Pi$, then $\mathfrak{n}^- = \mathrm{Lie}(U^-) = \bigoplus_{\beta \in \Delta_+ \setminus \Delta_0} \mathfrak{g}_{-\beta}$, where $\Delta_0 := \Delta \cap \mathrm{Span}_{\mathbb{Z}}(\Pi_0)$. So $U^- \times P \hookrightarrow G \Rightarrow U^- \hookrightarrow G/P$ (compare to Lemma in Sec 2.3 of Lec 13). Consider $Y = G \times^P F \xrightarrow{\pi} G/P$, $F := \{(\alpha, x) \mid \alpha|_{\mathfrak{l}} = \mu(x)\}$. Then $\pi^{-1}(U^-) \simeq U^- \times F$. Note that $\mathfrak{g}/\mathfrak{n} \simeq \mathfrak{l} \oplus \mathfrak{n}^-$ giving $F \xrightarrow{\sim} X_L \times (\mathfrak{n}^-)^*$ via $(\alpha, x) \mapsto (x, \alpha - \mu(x))$. So

$\mathrm{codim}_{\pi^{-1}(U^-)} \pi^{-1}(U^-)^{sing} \geqslant 4$. Since $G/P \simeq \bigcup_{g \in G} g U^-$ (open cover), we get an open cover $Y = \bigcup_{g \in G} g \pi^{-1}(U^-)$. It follows that $\mathrm{codim}_Y Y^{sing} \geqslant 4$, hence Y is terminal.

Now we need to prove that Y is $\mathbb{Q}$-factorial. For this we compute $\mathrm{Pic}(Y)$ & $\mathrm{Pic}(Y^{reg})$ and their inclusion. This is done in the next lemma, which will finish the proof, since $\mathrm{Pic}(X_2)=\{0\}$ & $\mathrm{Pic}(X_2^{reg})$ is finite, see Prop'n in Sec 1.2 of Lec 12 $\square$

Lemma: Assume G is simply connected. Then $\mathfrak{X}(L)\to \mathrm{Pic}(Y)$ & $\exists$ SES $0\to \mathrm{Pic}(Y)\to \mathrm{Pic}(Y^{reg})\to \mathrm{Pic}(X_2^{reg})\to 0$.

Proof: We have $\mathrm{Pic}(G/P)\xrightarrow{\sim}\mathfrak{X}(P)\xrightarrow{\sim}\mathfrak{X}(L)$, see Sec 1.2 in Lec 12, $\rightsquigarrow \pi^*: \mathfrak{X}(L)=\mathrm{Pic}(G/P)\to \mathrm{Pic}(Y)$. It's a split injection: $G/P \overset{\iota}{\hookrightarrow} Y$ via $gP\mapsto [g,(0,0)]$ & $\pi\circ\iota=id \Rightarrow \iota^*\circ\pi^*=id$. So π^*: $\mathfrak{X}(L)\hookrightarrow \mathrm{Pic}(Y^{reg})$. But $\mathrm{Pic}(G/P)=\mathrm{Cl}(G/P)$. Let $D_1,\dots D_k$ be codim 1 irreducible components of $(G/P)\setminus U^-$ (in fact, all components are of codim 1). We have (Hartschorne's book, Prop. 6.5) exact sequence $\mathrm{Span}_{\mathbb{Z}}(D_1,\dots D_k)\to \mathrm{Cl}(G/P)\to \mathrm{Cl}(U^-)=\{0\}$. The fibers of π are irreducible of the same dimension, so, for similar reasons we have the following exact sequence

$$\underset{\pi^*\mathrm{Pic}(G/P).}{\mathrm{Span}_{\mathbb{Z}}(\pi^{-1}(D_i))}\to \mathrm{Cl}(Y)\to \mathrm{Cl}(\underbrace{U^-\times \mathfrak{n}^{-*}}_{\text{affine space}}\times X_2)=\mathrm{Cl}(X_2)\to 0$$

Note also that $\mathrm{Pic}(U^-\times\mathfrak{n}^{-*}\times X_2)=\mathrm{Pic}(X_2)=\{0\}$. It follows that $\mathrm{Span}_{\mathbb{Z}}(\pi^{-1}(D_i)) = \pi^* \mathrm{Pic}(G/P)$ in $\mathrm{Pic}(Y^{reg})$ actually coincides w. the image of $\mathrm{Pic}(Y)$. The exact sequence in the statement follows. $\square$

Thx to the transitivity of induction (Sec 1.2 in Lec 15), & the minimality of L, we reduce Thm 1 to the case when $\tilde{\mathbb{O}}$ cannot be properly induced.

Def'n: $\tilde{\mathbb{O}}$ is birationally rigid if $\tilde{\mathbb{O}} = \mathrm{Ind}_L^G(\tilde{\mathbb{O}}_L) \Rightarrow L = G$.

Example: Let $G = Sp_{2n}$ & $\mathbb{O}$ be the orbit corresponding to $\tau = (2, 1^{2n-2})$. Then $\overline{\mathbb{O}}$ & its two-fold cover $\mathbb{C}^{2n}$ are birationally rigid (exercise).

The following theorem gives equivalent characterizations of birationally rigid covers.

Thm 2: TFAE:

(a) $X (= \mathrm{Spec}(\mathbb{C}[\tilde{\mathbb{O}}]))$ is $\mathbb{Q}$-factorial & terminal.

(b) $\tilde{\mathbb{O}}$ is birationally rigid.

(c) $H^2(Y^{reg}, \mathbb{C}) = \{0\}$, for a $\mathbb{Q}$-factorial terminalization $Y \to X$ (recall that $H^2(Y^{reg}, \mathbb{C}) = \mathfrak{h}_X$ is the Namikawa-Cartan space, Sec 2 of Lec 11).

1.3) Pic vs H^2.

In Algebraic Geometry, it's often easier to deal w. Pic than w. H^2 - but they are related via 1st Chern class, c_1.

Prop ([LMBM], Lemma 4.4.6) Let X be a conical symplectic singularity & Y is its $\mathbb{Q}$-factorial terminalization. Then the 1st Chern character map $c_1: \operatorname{Pic}(Y^{reg}) \longrightarrow H^2(Y^{reg}, \mathbb{Z})$ induces an isomorphism

$$\operatorname{Pic}(Y^{reg}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} H^2(Y^{reg}, \mathbb{C}).$$

This proposition will be used to prove (a)$\Leftrightarrow$(c) & also later.

Not-quite-a-proof:

We'll prove the corresponding statement in the complex analytic setting (and then one needs to algebrize, which is painful). Let $\mathcal{O}^{an}$ be the sheaf of analytic functions on Y^{reg} & $\mathrm{Pic}^{an}(Y^{reg})$ is the group of iso classes of complex analytic line bundles. Let $\mathcal{O}^{an,\times} \subset \mathcal{O}^{an}$ be the subsheaf of invertible functions. Then $\mathrm{Pic}^{an}(Y^{reg}) = H^1(\mathcal{O}^{an,\times})$. On the other hand, we have a SES of sheaves of abelian groups on Y^{reg}

$$0 \to \underline{\mathbb{Z}} \longrightarrow \mathcal{O}^{an} \xrightarrow{\exp} \mathcal{O}^{an,\times} \to 0$$

We have the corresponding long exact sequence in cohomology of which the relevant piece is:

$$H^1(\mathcal{O}^{an}) \to H^1(\mathcal{O}^{an,\times}) \to H^2(Y^{reg}, \mathbb{Z}) \to H^2(\mathcal{O}^{an})$$

We have seen, Sec 2 of Lec 12, that $H^1(\mathcal{O}_{Y^{reg}}) = H^2(\mathcal{O}_{Y^{reg}}) = 0$. For the same reason, their analytic counterparts vanish. And we get $H^1(\mathcal{O}^{an,\times}) \xrightarrow{\sim} H^2(Y^{reg}, \mathbb{Z})$ $\square$

1.4) a) $\Leftrightarrow$ c):

a) $\Rightarrow$ c): We know $\mathrm{Pic}(X) = \{0\}$ (Sec 1.2 in Lec 12) $\Rightarrow$ [X is $\mathbb{Q}$-factorial] $\mathrm{Pic}(X^{reg})$ is finite $\Rightarrow$ [Prop in Sec 1.3] $H^2(X^{reg}, \mathbb{C}) = \{0\}$.

c) $\Rightarrow$ a): Pick a very ample line bundle $\mathcal{L}$ on Y. $\forall$ $\mathbb{C}[X]$-module generators $s_1, \dots s_k$ of $\Gamma(Y, \mathcal{L})$, the morphism $Y \xrightarrow{\rho} X$ decomposes as

$$Y \hookrightarrow X\times \mathbb{P}^{k-1} \longrightarrow X, \qquad (1)$$

where the 1st map is $y\mapsto (\rho(y), [s_1(y):\ldots:s_k(y)])$. We can replace $\mathcal{L}$ w. a multiple. By Prop. in Sec 1.3, $\mathrm{Pic}(Y^{reg})$ is finite. So we can assume $\mathcal{L}$ is trivial $\rightsquigarrow$ can take $k=1$, $s_1=1$. (1) $\Rightarrow Y \xrightarrow{\sim} X$. $\square$

Rem: This applies to any conical symplectic singularity X.

1.5) a) $\Rightarrow$ b). If $\tilde{\mathbb{O}}$ is not birationally rigid, $\tilde{\mathbb{O}}=\mathrm{Ind}_L^G(\tilde{\mathbb{O}}_L)$, then $Y=\mathrm{Ind}_P^G(X_L)$ is a partial Poisson resolution of X, nontrivial b/c $G/P\hookrightarrow Y$. So X is not maximal & by Thm in Sec 1.3 of Lec 12, X is not $\mathbb{Q}$-factorial terminal. Contradiction.

1.6) b) $\Rightarrow$ a), preparation.

This is the hardest part of the proof. It's based on a result of Namikawa, that we'll explain in this lecture.

Definition: Let X be a conical symplectic singularity & B be a fin. gen'd positively graded algebra. A graded Poisson deformation of X over $\mathrm{Spec}(B)$ is a Poisson scheme $X_{\mathrm{Spec}(B)}$

over $\mathrm{Spec}(B)$ (for $p \in \mathrm{Spec}(B)$, set $X_p := \{p\} \times_{\mathrm{Spec}(B)} X_{\mathrm{Spec}(B)}$) s.t.

(0) $X_{\mathrm{Spec}(B)}$ is flat over $\mathrm{Spec}(B)$.

(1) $\mathbb{C}^\times \curvearrowright X_{\mathrm{Spec}(B)}$ s.t. $\{\cdot,\cdot\}$ on $\mathbb{C}[X_{\mathrm{Spec}(B)}]$ has $\deg = -d$ (same as on $\mathbb{C}[X]$).

(2) A $\mathbb{C}^\times$-equivariant Poisson isomorphism $X \xrightarrow{\sim} X_0$.

Rem: From here we get a family of filtered Poisson deformations of $\mathbb{C}[X]$ indexed by pts of $\mathrm{Spec}(B)$: $p \in \mathrm{Spec}(B)$ $\rightsquigarrow \mathbb{C}[X_p]$. Conversely, for a filtered Poisson deformation $\mathcal{A}^\circ$ of $\mathbb{C}[X]$, take $B := \mathbb{C}[\hbar]$ & set $X_{\mathbb{A}^1} := \mathrm{Spec}(R_\hbar(\mathcal{A}^\circ))$.

Thm ([N2a], [N2b]): Set $\mathfrak{h}_X := H^2(Y^{reg}, \mathbb{C})$ for a $\mathbb{Q}$-factorial terminalization Y of X. Then $\exists$ reflection group $W_X \subset GL(\mathfrak{h}_X)$ & graded Poisson deformation $X_{\mathfrak{h}_X/W_X}$ w. the following universality property: $\forall$ B & $X_{\mathrm{Spec}(B)}$ as in Def'n $\exists$ graded algebra homomorphism $\mathbb{C}[\mathfrak{h}_X]^{W_X} \to B$ & isomorphism of graded Poisson deformations (i.e. Poisson isomorphism of schemes over $\mathrm{Spec}(B)$ intertwining (1)&(2)) $\mathrm{Spec}(B) \times_{\mathfrak{h}_X/W_X} X_{\mathfrak{h}_X/W_X} \longrightarrow X_B$.
The former is unique.

Combining Thm with the previous remark we get the following corollary (part of the main Thm in Sec 2 of Lec 11) whose proof is an exercise.

Corollary: The filtered Poisson deformations of $\mathbb{C}[X]$ are classified (up to isomorphism) by the pts of $\mathfrak{h}_X/W_X$.

Example: Let $X=\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}^*$. Then as mentioned in Sec 2 of Lec 11, $\mathfrak{h}_X=\mathfrak{h}^*$, $W_X=W$. We have $X_{\mathfrak{h}^*/W}=\mathfrak{g}^*$, where the morphism $\mathfrak{g}^*\to\mathfrak{g}^*/\!/G\xrightarrow{\sim}\mathfrak{h}^*/W$ is the quotient morphism for $G\curvearrowright\mathfrak{g}^*$. This morphism is flat: the source & target are smooth & all fibers have the same dimension.