Lecture 17.

1) $\mathbb{Q}$-factorial terminalizations from induction, II.

Ref: [B], Ch. 1; [L1], Sec 4;

1.0) Reminder

We work to prove the following theorem, Sec 1.1 of Lec 16.

Thm 1: Let L be a minimal Levi in G s.t $\widetilde{\mathbb{O}} = \operatorname{Ind}_L^G(\widetilde{\mathbb{O}}_L)$. Then $Y = \operatorname{Ind}_P^G(X_L)$ is a $\mathbb{Q}$-factorial terminalization of X ($= \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}]$).

We have reduced this to proving:

Thm 2: If $\widetilde{\mathbb{O}}$ is birationally rigid (i.e. cannot be induced from a cover in a proper Levi). Then X is $\mathbb{Q}$-factorial & terminal, equivalently, $\mathfrak{h}_X = \{0\}$.

A key to proving Thm 2 is Namikawa's result on the existence of a universal graded Poisson deformation $X_{\mathfrak{h}_X/W_X}$

of X over $\mathfrak{h}_X/W_X$.

1.1) Lifting Hamiltonian actions.

Our 1st step in the proof is the following general result that allows to extend Hamiltonian actions to deformations.

Proposition: Let X be a conical symplectic singularity, G is a simply connected semisimple group w. Hamiltonian action on X that commutes w. the $\mathbb{C}^\times$-action. Let X_Z be a graded Poisson deformation of X ($Z = \operatorname{Spec}(B)$ from Sec 1.6 of Lec 16). Then the G-action extends to a Hamiltonian G-action on X_Z that commutes w. $\mathbb{C}^\times$ & makes $X_Z \longrightarrow Z$ G-invariant.

Proof: We can reduce to the case when $\mathfrak{g} \hookrightarrow \mathbb{C}[X]$ (exercise). G & $\mathbb{C}^\times$ commute, so we see that $\{\mathfrak{g},\cdot\}$ preserves the grading, hence $\mathfrak{g} \hookrightarrow \mathbb{C}[X]_d$ (where $d = -\deg\{\cdot,\cdot\}$). Note that $\mathbb{C}[X_Z]_d \twoheadrightarrow \mathbb{C}[X]_d$. We claim that the kernel, K, is a nilpotent Lie algebra.

Indeed, an element $f \in K$ is of the form $\sum_i b_i f_i$ w. $b_i \in \mathbb{C}[Z]_{d_i}$, $0 < d_i < d$, $f_i \in \mathbb{C}[X_Z]_{d-d_i}$ (we can take a graded basis in $\mathbb{C}[X]$, say $\underline{f}_i, i\in I$, then lift them to homogeneous $f_i \in \mathbb{C}[X]_Z$, these elements generate the $\mathbb{C}[Z]$-module $\mathbb{C}[X_Z]$ by graded Nakayama and we take any decomposition of f). The bounds on the degrees are b/c all gradings are positive.

The elements b_i are Poisson central in $\mathbb{C}[X_Z]$, from the definition. So $\{\sum_i b_i f_i, \sum_j b_j' f_j'\} = \sum_{i,j} b_i b_j' \{f_i, f_j'\}$. Note that $\min\{\deg b_i b_j'\} > \min\{\deg b_i\}$. From here we deduce that the $d+1$-fold brackets vanish on K.

Also note that $\dim K < \infty$. Let $\tilde{\mathfrak{g}}$ be the preimage of $\mathfrak{g} \subset \mathbb{C}[X]_d$ in $\mathbb{C}[X_Z]_d$ so that we have a Lie algebra SES

$$0 \to K \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

Since K is nilpotent, Levi's thm ([B], Ch. 1, Sec 6.8) shows that the SES splits and so we have an embedding $\mathfrak{g} \hookrightarrow \mathbb{C}[X_Z]_d$ lifting $\mathfrak{g} \hookrightarrow \mathbb{C}[X]_d$. The representation of $\mathfrak{g}$ in $\mathbb{C}[X_Z]$ by taking $\{\cdot,\cdot\}$ preserves $\mathbb{C}[X_Z]_{\leq k}$ $\forall k$. These spaces are finite dimensional again by the positivity of the grading. So the $\mathfrak{g}$-action of $\mathbb{C}[X_Z]$ integrates to G. Since $\mathfrak{g}$ acts by derivations, G acts by automorphisms giving $G \curvearrowright X_Z$. To show that this action is Hamiltonian & finish the proof is left as an exercise. $\square$

Rem: The condition that G is simply connected can be removed (exercise).

1.2) Structure of deformation X_λ.

Let $\tilde{\mathbb{O}}$ be a G-equivariant cover of a nilpotent orbit in $\mathfrak{g}$. Suppose $\mathfrak{h}_X \neq \{0\}$. We pick a nonzero element $\lambda \in \mathfrak{h}_X$. Consider the natural morphism $\mathbb{C}\lambda \longrightarrow \mathfrak{h}_X/W_X$, it's $\mathbb{C}^\times$-equivariant. So, we can form the graded deformation $X_{\mathbb{C}\lambda}$ of $X = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$. By Proposition in Sec 1.1, we have a Hamiltonian action of G on $X_{\mathbb{C}\lambda}$ commuting w. the $\mathbb{C}^\times$-action & the moment map $\mu: X_{\mathbb{C}\lambda} \longrightarrow \mathfrak{g}^*$ is $\mathbb{C}^\times$-equivariant, where $\mathbb{C}^\times$ acts on $\mathfrak{g}^*$ by $t.\alpha = t^2\alpha$. Let $\eta: X_{\mathbb{C}\lambda} \to \mathbb{C}\lambda$ be the natural morphism. Our next goal is to describe X_λ.

Lemma 1) $\forall z \neq 0$, $X_{z\lambda}$ has a unique open G-orbit of the same dimension as $\dim \tilde{\mathbb{O}}$, denote it by $\tilde{\mathbb{O}}_\lambda$.

2) $(\mu,\eta): X_{\mathbb{C}\lambda} \to \mathfrak{g}^* \times \mathbb{C}\lambda$ is finite, hence $\mu: X_\lambda \to \mathfrak{g}^*$ is finite; moreover, $\mu(\tilde{\mathbb{O}}_\lambda) =: \mathbb{O}_\lambda$ is an orbit & $\mu: \tilde{\mathbb{O}}_\lambda \to \mathbb{O}_\lambda$ is a cover.

3) $\mathbb{O}_\lambda$ is not nilpotent.

Proof: 1): repeats that of Case 3 in Sec 2.2 of Lec 14: use that $\dim X = \dim X_{z\lambda}$ as the deformation is flat & that the locus of points in $X_{\mathbb{C}\lambda}$ w. maximal orbit dimension is open, & that $X_{z\lambda}$ is irreducible, equiv. $\mathbb{C}[X_{z\lambda}]$ is a domain - the latter follows from $\operatorname{gr} \mathbb{C}[X_{z\lambda}] \xrightarrow{\sim} \mathbb{C}[X]$.

2): $\mu: X \to \mathfrak{g}^*$ is finite $\Rightarrow$ [graded Nakayama, exercise] $(\mu,\eta): X_{\mathbb{C}\lambda} \to \mathfrak{g}^* \times \mathbb{C}\lambda$ is finite $\Rightarrow \mu: X_\lambda \to \mathfrak{g}^*$ is finite.

And $\mu(\tilde{\mathbb{O}}_\lambda)$ is a G-orbit & $\tilde{\mu}: \tilde{\mathbb{O}}_\lambda \to \mu(\tilde{\mathbb{O}}_\lambda)$ is a cover b/c $\tilde{\mathbb{O}}_\lambda$ carries a transitive Hamiltonian action, see Sec 1.2 of Lec 3.

3) Assume $\mathbb{O}_\lambda$ is nilpotent $\Rightarrow \mathbb{O}_{z\lambda} = z.\mathbb{O}_\lambda = \mathbb{O}_\lambda \Rightarrow \mathbb{O} \subset \mu(X_{\mathbb{C}\lambda}) = \overline{\mathbb{O}}_\lambda$; $\dim \mathbb{O} = \dim X = \dim X_\lambda = \dim \mathbb{O}_\lambda \Rightarrow \mathbb{O} = \mathbb{O}_\lambda$. By 2), the morphism $X_{\mathbb{C}\lambda} \xrightarrow{(\mu,\eta)} \overline{\mathbb{O}} \times \mathbb{C}\lambda$ is finite; it's $G \times \mathbb{C}^\times$-equivariant & Poisson.

HW problem 1: We have a commutative diagram

$$\begin{tikzcd} X\times\mathbb{C}\lambda \arrow[rr] \arrow[dr] & & X_{\mathbb{C}\lambda} \arrow[dl] \\ & \overline{\mathbb{O}}\times\mathbb{C}\lambda & \end{tikzcd}$$

where the horizontal arrow is a $G\times\mathbb{C}^\times$-equivariant Poisson isomorphism.

So $X_{\mathbb{C}\lambda}\simeq\mathbb{C}\lambda\times_{\mathfrak{z}_X/W_X}X_{\mathfrak{z}_X/W_X}$ for the <u>zero</u> map $\mathbb{C}\lambda\longrightarrow\mathfrak{z}_X/W_X$. But the map $\mathbb{C}\lambda\to\mathfrak{z}_X/W_X$ s.t. $\exists$ isom'm $X_{\mathbb{C}\lambda}\xrightarrow{\sim}\mathbb{C}\lambda\times_{\mathfrak{z}_X/W_X}X_{\mathfrak{z}_X/W_X}$ is unique by Namikawa's thm in Sec 1.6 of Lec 16. Since $\lambda\neq 0$, we arrive at a contradiction. $\square$

We proceed to giving an explicit description of X_λ.

HW problem 2: The inclusion $\widetilde{\mathbb{O}}_\lambda\hookrightarrow X_\lambda$ yields $\mathbb{C}[X_\lambda]\xrightarrow{\sim}\mathbb{C}[\widetilde{\mathbb{O}}_\lambda]$.

We'll need an equivalent description. We use a construction from Sec 1.3 in Lec 15. Take $\xi\in\mathbb{O}_\lambda$ and let $L:=Z_G(\xi_s)$ & $\widetilde{\mathbb{O}}_L:=\mu^{-1}(\xi_s+\mathbb{O}_L)$, where $\mathbb{O}_L\subset\mathfrak{l}^*$ is the nilpotent orbit corresponding to $\mathbb{O}_\lambda$.

Recall, Sec 1.3 of Lec 15, that $\operatorname{Spec}\mathbb{C}[\widetilde{\mathbb{O}}_\lambda] \xrightarrow{\sim} Y_{\xi_S} =$ $= G\times^L(\xi_S\times X_L)\ (= G\times^P\{(\alpha,x)\in(\mathfrak{g}/\mathfrak{h})^*\times X_L \mid \alpha|_{\mathfrak{l}} = \mu(x)+\xi_S\})$. This a G-equivariant Poisson isomorphism intertwining the moment maps. So we have $X_\lambda \xrightarrow{\sim} Y_{\xi_S}$ w. the same properties. Here's the description of X_λ that we need:

$$Y_{\xi_S} \xrightarrow{\sim} X_\lambda \tag{1}$$

1.3) Identification of $\widetilde{\mathbb{O}}$ w. $\operatorname{Ind}_L^G(\widetilde{\mathbb{O}}_L)$.

Finally, we identify $\widetilde{\mathbb{O}}$ w. $\operatorname{Ind}_L^G(\widetilde{\mathbb{O}}_L)$. Identify $\mathbb{C}\xi_S$ w. $\mathbb{C}\lambda$ by sending ξ_S to λ. Thx to (1), we have a $G\times\mathbb{C}^\times$-equivariant Poisson morphism that intertwines the maps to $\mathfrak{g}^*\times\mathbb{C}\lambda$

$$\mathbb{C}^\times\times_{\mathbb{C}\lambda} Y_{\mathbb{C}\lambda} \longrightarrow X_{\mathbb{C}\lambda} \tag{2}$$

More precisely, we use $\mathbb{C}^\times\times Y_{\xi_S} \xrightarrow{\sim} \mathbb{C}^\times\times X_\lambda$ & the actions of $\mathbb{C}^\times$ (and G) on $Y_{\mathbb{C}\lambda}, X_{\mathbb{C}\lambda}$ to deduce $\mathbb{C}^\times_{\mathbb{C}\lambda} Y_{\mathbb{C}\lambda} \xrightarrow{\sim} \mathbb{C}^\times_{\mathbb{C}\lambda} X_{\mathbb{C}\lambda}$, yielding (2) (there are some technicalities swept under rug).

By 2) of Lemma, $X_{\mathbb{C}\lambda}\longrightarrow\mathfrak{g}^*\times\mathbb{C}\lambda$ is finite, hence proper. Apply the valuative criterium of properness (Hartschorne's book, Chapter 2, Sec. 4) to the rational map

$$Y_{\mathbb{C}\lambda} \dashrightarrow X_{\mathbb{C}\lambda} \qquad (3)$$

We see that (3) extends (uniquely) to a morphism $\rho: Y^0_{\mathbb{C}\lambda} \longrightarrow X_{\mathbb{C}\lambda}$, where $Y^0_{\mathbb{C}\lambda} \subset Y_{\mathbb{C}\lambda}$ is the domain of def'n of (3), it's open & the complement has codim $\geqslant 2$. G-equivariance of (2) $\Rightarrow Y^0_{\mathbb{C}\lambda}$ is G-stable $\Rightarrow \tilde{\mathbb{O}}' := \mathrm{Ind}_L^G(\tilde{\mathbb{O}}_L) \subset Y^0_{\mathbb{C}\lambda}$. Also ρ intertwines the morphisms to $\mathfrak{g}^* \times \mathbb{C}\lambda$. Both $\tilde{\mathbb{O}}, \tilde{\mathbb{O}}'$ are covers of orbits of $\dim = \dim \mathbb{O}_\lambda$ via μ. So $\dim \tilde{\mathbb{O}}' = \dim \tilde{\mathbb{O}}$.

Hence $\rho: \tilde{\mathbb{O}}' \longrightarrow \tilde{\mathbb{O}}$ is a cover $\rightsquigarrow \mathbb{C}[\tilde{\mathbb{O}}] \underset{G}{\hookrightarrow} \mathbb{C}[\tilde{\mathbb{O}}']$. On the other hand, we have $\mathbb{C}[\tilde{\mathbb{O}}] \simeq \mathrm{gr}\, \mathbb{C}[X_\lambda]$ (Sec. 1.6 of Lec 16) & $\mathbb{C}[\tilde{\mathbb{O}}'] \simeq$ [Sec 2 of Lec 14] $\simeq \mathrm{gr}\, \mathbb{C}[Y_\lambda]$ These are G-linear isomorphisms. By (1), $\mathbb{C}[Y_\lambda] \simeq \mathbb{C}[X_\lambda]$ So $\mathbb{C}[\tilde{\mathbb{O}}] \simeq_G \mathbb{C}[\tilde{\mathbb{O}}']$. Note that $G \curvearrowright \mathbb{C}[\tilde{\mathbb{O}}], \mathbb{C}[\tilde{\mathbb{O}}']$ are rational. Now $\tilde{\mathbb{O}} \simeq \tilde{\mathbb{O}}'$ follows from

Claim: Let $H \subset G$ be an algebraic subgroup. Then $\forall$ fin. dim. G-irrep V have $\mathrm{Hom}_G(V, \mathbb{C}[G/H]) \simeq (V^*)^H$, so is finite dimensional.

Sketch of proof: By the algebraic version of the Peter-Weyl thm, $\mathbb{C}[G] \xrightarrow[G\times G]{\sim} \bigoplus_{U} U\otimes U^*$, where the sum is taken over all G-irreps U. Then $\mathbb{C}[G/H] \xrightarrow{\sim} \mathbb{C}[G]^H \xrightarrow{\sim} \bigoplus_{U} U\otimes (U^*)^H$ and the claim follows. $\square$

Conclusion: Our assumption was $\mathfrak{z}_x \neq \{0\}$. We see that $\widetilde{\mathbb{O}}$ is not birationally rigid. So, if $\widetilde{\mathbb{O}}$ is birationally rigid, then $\mathfrak{z}_x = \{0\}$. This finishes the proof of Thm 2.

Remark: The morphism $Y^{\circ}_{\mathbb{C}\lambda} \longrightarrow X_{\mathbb{C}\lambda}$ extends to $Y_{\mathbb{C}\lambda} \longrightarrow X_{\mathbb{C}\lambda}$ b/c $\mathbb{C}[Y_{\mathbb{C}\lambda}] = \mathbb{C}[Y^{\circ}_{\mathbb{C}\lambda}]$ (by Hartogs) & $X_{\mathbb{C}\lambda}$ is affine. Once we know that $\widetilde{\mathbb{O}}' \xrightarrow{\sim} \mathbb{O}$, we see that $Y^{\circ}_{\mathbb{C}x} \hookrightarrow X_{\mathbb{C}\lambda}$ (as sets), in fact, this is an open embedding. From here we deduce $X_{\mathbb{C}\lambda} \simeq \operatorname{Spec} \mathbb{C}[Y_{\mathbb{C}\lambda}]$.