Lecture 18

1) Wrap-up on Poisson deformations & $\mathbb{Q}$-factorial terminalizations.
2) Sheaves of twisted differential operators.

Ref: [LMBM]; [G], Sec 2.

1.1) Construction of universal graded Poisson deformation.

Let $G, \mathfrak{g}, \widetilde{\mathbb{O}}, X, L, P, \widetilde{\mathbb{O}}_L, X_L$ have the same meaning as before. Let $\mathfrak{z} := (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. We have seen in Lecs 16 & 17 that if L is minimal w. $\widetilde{\mathbb{O}} = \operatorname{Ind}_L^G(\widetilde{\mathbb{O}}_L)$, then $Y = \operatorname{Ind}_P^G(X_L)$ is a $\mathbb{Q}$-factorial terminalization of X. We have also seen in Sec. 2 of Lec 15, for $\lambda \in \mathfrak{z}$, $\mathbb{C}[Y_\lambda]$ is a filtered Poisson deformation of $\mathbb{C}[X]$. In the course of the proof we have seen that $\operatorname{Spec} \mathbb{C}[Y_{\mathbb{C}\lambda}]$ is a graded Poisson deformation of X. The following can be proved along the same lines.

Theorem: $X_{\mathfrak{z}} := \operatorname{Spec} \mathbb{C}[Y_{\mathfrak{z}}]$ is a graded Poisson deformation of X over $\mathfrak{z}$.

So, $X_{\mathfrak{z}} \xrightarrow{\sim} \mathfrak{z} \times_{\mathfrak{h}_X/W_X} X_{\mathfrak{h}_X/W_X}$ for unique $\mathfrak{z} \to \mathfrak{h}_X/W_X$, by Namikawa's thm from Sec 1.6 in Lec 16. We want to determine $\mathfrak{z} \to \mathfrak{h}_X/W_X$.

Note that by Lemma in Sec 1.2 of Lec 16, we have $\mathfrak{X}(L) \xrightarrow{\sim} \mathrm{Pic}(Y) \hookrightarrow \mathrm{Pic}(Y^{reg})$ (w. finite cokernel). And by Sec 1.3 in Lec 16, $\mathfrak{h}_X = H^2(Y^{reg}, \mathbb{C}) \xrightarrow{\sim} \mathrm{Pic}(Y^{reg}) \otimes_{\mathbb{Z}} \mathbb{C}$. We conclude that $\mathfrak{z} = \mathfrak{X}(L) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathrm{Pic}(Y^{reg}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathfrak{h}_X$.

Fact: This is the quotient morphism for $W_X \curvearrowright \mathfrak{z} = \mathfrak{h}_X$.

Explanation: For any $\mathbb{Q}$-factorial terminalization Y of a conical symplectic singularity X, Namikawa established a "universal graded deformation" $Y_{\mathfrak{h}_X}$, a Poisson scheme over $\mathfrak{h}_X$ w. 0-fiber Y. He checked that $W_X \curvearrowright X_{\mathfrak{h}_X} := \mathrm{Spec}\, \mathbb{C}[Y_{\mathfrak{h}_X}]$ & $X_{\mathfrak{h}_X/W_X} := X_{\mathfrak{h}_X}/W_X$. By the universality, $\exists!$ <u>linear</u> $\mathfrak{z} \to \mathfrak{h}_X$ s.t. $Y_{\mathfrak{z}} \xrightarrow{\sim} \mathfrak{z} \times_{\mathfrak{h}_X} Y_{\mathfrak{h}_X}$. And one can show that this map is injective, hence an isomorphism, see Proposition 7.2.2 in [LMBM]. $\square$

In particular, $X_{\mathfrak{z}}$ is independent of the choice of P (exercise).

Example: Let $X=\mathcal{N}$. Then $Y_{\mathfrak{z}}:=G\times^B(\mathfrak{g}/\mathfrak{n})^*$, Sec 1.1 of Lec 14.
We have commutative diagram
$$\begin{tikzcd} Y_{\mathfrak{z}} \arrow[r] \arrow[d] & \mathfrak{g}^* \arrow[d] \\ \mathfrak{h}^* \arrow[r] & \mathfrak{h}^*/W \end{tikzcd}$$

$\rightsquigarrow Y_{\mathfrak{z}}\to \mathfrak{g}^*\times_{\mathfrak{h}^*/W}\mathfrak{h}^*=$ [exercise] $=X_{\mathfrak{z}}$; $W_X=W$ acts on $\mathfrak{h}^*$-factor &
$X_{\mathfrak{z}}/W=\mathfrak{g}^*\times_{\mathfrak{h}^*/W}\mathfrak{h}^*/W=\mathfrak{g}^*=$ [Ex in Sec 1.6 of Lec 16] $=X_{\mathfrak{h}^*/W}$.

1.2) Classification of $\mathbb{Q}$-factorial terminalizations.

Fact: 1) The pair $(M,\tilde{\mathbb{O}}_M)$ s.t. $\mathrm{Ind}_Q^G(X)$ is a $\mathbb{Q}$-factorial terminalization of X is determined uniquely from X.

2) $\forall$ $\mathbb{Q}$-factorial terminalization of X has the form $\mathrm{Ind}^G(X_M)$ for some parabolic subgroup $Q\subset G$ w. Levi M.

Explanation: 1) We know that $(\mathfrak{m}/[\mathfrak{m},\mathfrak{m}])^*\xrightarrow{\sim}\mathfrak{h}_X$. From here one deduces that M must be minimal among all Levi's s.t. $\tilde{\mathbb{O}}$ is induced from a cover for M. For such M, any cover $\tilde{\mathbb{O}}_M$ s.t. $\tilde{\mathbb{O}}=\mathrm{Ind}_M^G(\tilde{\mathbb{O}}_M)$ is birationally rigid. Details are left as an exercise.

Consider a graded Poisson deformation X_Z, where $\mathbb{C}[Z]$ is a domain. One can follow the argument in Sec 1.1 of Lec 17 to show that for different choices of lift of the moment map from X to X_Z there's an automorphism (of a graded Poisson deformation) of X_Z intertwining them: it belongs to $\exp\{K,\cdot\}$, where K is as in the proof there, this is based on Mal'cev's thm ([B] Ch. 1, Sec 6.8), left as exercise.

Pick a Zariski generic element $p\in Z$ and form $(M,\widetilde{\mathbb{O}}_M)$ of a Levi M & an M-equiv't cover $\widetilde{\mathbb{O}}_M$ of a nilpotent orbit in $\mathfrak{m}^*$ as in the end of Sec 1.2 of Lec 17: e.g. $M=Z_G(\xi_s)$, where $\xi=\mu(x)$ for Zariski generic $x\in X_p$. The pair $(M,\widetilde{\mathbb{O}}_M)$ is independent of the choice of x. This is a relatively technical & relatively standard algebro-geometric result based on the observation that there are only finitely many choices of $(M,\widetilde{\mathbb{O}}_M)$ up to G-conjugacy. For non-generic $p\in Z$, the corresponding M' contains a conjugate of M. Note that $\widetilde{\mathbb{O}}=\operatorname{Ind}_{M'}^G(\widetilde{\mathbb{O}}_{M'})$.

If we apply this to $Z=\mathfrak{z}$, $X_Z=X_{\mathfrak{z}}$ from Thm in Sec 1.1, then $M=L$. And the smallest possible M comes from the universal deformation $X_{\mathfrak{z}_X/W_X}$. The corresponding pair $(M,\widetilde{\mathbb{O}}_M)$ is what we need. Details are left as exercise.

2) By some standard birational geometry, to show that there are no other $\mathbb{Q}$-factorial terminalizations amounts to showing that the ample cones of all $Ind_Q^G(X_M)$ cover $\mathfrak{h}_{X,\mathbb{R}} = H^2(Y^{reg}, \mathbb{R})$ (for fixed Y – these spaces are identified for different choices of Y). By Sec 1.2 of Lec 16), $Pic(Ind_Q^G(X_M)) \simeq \mathfrak{X}(M)$. Under this identification ample (relative to $Ind_Q^G(X_M) \to X$) line bundles correspond to characters of M that are strictly dominant for P so the ample cone in $\mathfrak{h}_{X,\mathbb{R}}$ is the cone of real dominant wts. These cones (for various Q) cover $\mathfrak{h}_{X,\mathbb{R}}$, yielding our result. $\square$

2) Sheaves of twisted differential operators.

2.0) Motivation.

We now proceed to the quantum part of the story. Our goal is to produce filtered quantizations of $\mathbb{C}[\tilde{O}]$. This will be done by quantizing $Y = Ind_P^G(X_L)$ – we'll explain what we mean by this later – and taking the global sections. The quantizations of Y will be constructed by quantum Hamiltonian reduction.

Note that some of Y's are of the form $T^*(G/P)$ for a parabolic subgroup $P \subset G$. There is a general classification of filtered quantizations of T^*Y_0 (for smooth Y_0): the quantizations are sheaves of twisted differential operators. This is what we are going to explain now.

2.1) TDO, affine case.

Let Y_0 be a smooth affine variety.

Definition: By an algebra of twisted differential operators on Y_0 we mean a filtered quantization of $\mathbb{C}[T^*Y_0]$

Recall that $\mathbb{C}[T^*Y] = S_A(V)$ w. $A = \mathbb{C}[Y_0]$, $V = \mathrm{Vect}(Y_0)$. So we have a SES $0 \to A \to \mathcal{D}_{\leq 1} \to V \to 0$ of A-modules that splits b/c V is a projective A-module, but the splitting is not unique, if we fix one, say $\iota: V \to \mathcal{D}_{\leq 1}$, then the others take the form $\iota + \alpha$ w. $\alpha: V \to A$, A-linear map i.e. a 1-form on Y_0. Note that A & $\iota(V)$ generate $\mathcal{D}$ as an algebra b/c A & V generate $S_A(V) = \mathrm{gr}\,\mathcal{D}$.

Recall (Lec 2) that the Poisson bracket on $\mathbb{F}[T^*Y_0]$ is recovered from:

$$\{f_1, f_2\} = 0$$

(1) $$\{\xi_1, f_1\} = \xi_1 . f_1$$

$$\{\xi_1, \xi_2\} = [\xi_1, \xi_2]_V, \quad f_1, f_2 \in A, \ \xi_1, \xi_2 \in V.$$

bracket in V.

It follows that in $\mathcal{D}$ we have:

$$[f_1, f_2]_{\mathcal{D}} = 0$$

def'n of ι.

(2) $$f_1 \iota(\xi_1) = \iota(f_1 \xi_1), \quad [\iota(\xi_1), f_1]_{\mathcal{D}} = \xi_1 . f_1$$

$$[\iota(\xi_1), \iota(\xi_2)]_{\mathcal{D}} = \iota([\xi_1, \xi_2]_V) + \beta(\xi_1, \xi_2),$$

where β is a 2-form.

For example, for $[\iota(\xi_1), f_1]_{\mathcal{D}} = \xi_1 . f_1$ we observe that since $\mathcal{D}$ is a filtered quantization of $\mathbb{F}[Y]$, the difference has degree -1 (the expected degree is 0 but the deg 0 term is $\{\xi_1, f_1\} - \xi_1 . f_1 = 0$. Since $\mathcal{D}_{\leq -1} = \{0\}$, the equality holds.

Important Exercise: Hint: Cartan Magic formula.

1) The Jacobi identity for $\iota(\xi_1), \iota(\xi_2), \iota(\xi_3)$ is equivalent to $d\beta = 0$.

2) For 1-form α $\exists!$ filt. quantization isomorphism

$\mathcal{D}_\beta \xrightarrow{\sim} \mathcal{D}_{\beta+d\alpha}$ w. $f \mapsto f$, $\iota(\xi) \mapsto \iota(\xi) + \langle \alpha, \xi \rangle$.

This gives rise to a map from the set of isomorphism classes of TDO (as quantizations) to $H^2_{DR}(Y_0) = H^2(Y_0, \mathbb{C})$.

Proposition: This map is an isomorphism.

Sketch of proof:

Let $\mathcal{D}_\beta$ denote the algebre generated by A, V w. relations (2). Then we have a natural graded algebra epimorphism $S_A(V) \twoheadrightarrow \operatorname{gr} \mathcal{D}_\beta$. We need to show it's an iso.

Pick $y \in Y_0$ and let $\hat{A}$ denote the completion of A at the maximal ideal corresponding to y. Note that $\hat{A} \otimes_A S_A(V) \xrightarrow{\sim} S_{\hat{A}}(\hat{V})$, where $\hat{V} = \hat{A} \otimes_A V$. Also $\hat{\mathcal{D}}_\beta := \hat{A} \otimes_A \mathcal{D}_\beta$ is naturally an algebra: it's generated by $\hat{A}$ & $\hat{V}$ w. relations (2). But it's independent of β up to quantization iso (see (2) of Important Exercise) by a formal version of Poincare lemma. Next, we know that $S_A(V) \xrightarrow{\sim} \operatorname{gr} \mathcal{D}_0$ ($\mathcal{D}_0$ is the usual DO) so $\hat{A} \otimes_A S_A(V) \longrightarrow \hat{A} \otimes_A \operatorname{gr} \mathcal{D}_\beta$ $\forall y$. This implies $S_A(V) \xrightarrow{\sim} \operatorname{gr} \mathcal{D}_\beta$, details are left as an exercise. $\square$

2.2) TDO, general case.

Now assume Y_0 is not affine (but still smooth). Consider the projection $\pi: Y = T^*Y_0 \twoheadrightarrow Y_0$. Then $\pi_* \mathcal{O}_Y$ is a graded sheaf of Poisson algebras on Y_0. We can talk about its filtered quantizations. By definition, this is a sheaf $\mathcal{D}$ of filtered algebras on Y_0 w. filtration $\mathcal{D} = \bigcup_{i \geq 0} \mathcal{D}_{\leq i}$ by $\mathcal{O}_{Y_0}$-coherent modules together w. a graded Poisson algebra iso $\pi_* \mathcal{O}_Y \xrightarrow{\sim} \operatorname{gr} \mathcal{D}$. Such a quantization is called a <u>sheaf of twisted differential operators</u>.

Let's explain how to classify sheaves of TDO. Pick an open affine cover $Y_0 = \bigcup_i Y_i^0$. From $\mathcal{D}(Y_i^0)$ we can read a closed 2-form $\beta_i \in \Gamma(Y_i^0, \Omega^2_{Y^0})$ (defined up to adding an exact form). Then we have isomorphisms of filtered quantizations $\mathcal{D}_{\beta_i}(Y_{ij}^0) \xrightarrow[\varphi_{ji}]{} \mathcal{D}_{\beta_j}(Y_{ij}^0)$ (w. $Y_{ij}^0 = Y_i^0 \cap Y_j^0$), they give rise to $\alpha_{ji} \in \Gamma(Y_{ij}^0, \Omega^1_{Y^0})$ w. $\varphi_{ji}: f \mapsto f, \xi \mapsto \xi + \langle \alpha_{ij}, \xi \rangle$ s.t. $\beta_j - \beta_i = d\alpha_{ji}$ (on Y_{ij}^0). The isomorphisms φ_{ji} satisfy the cocycle condition $\varphi_{ik} \circ \varphi_{kj} \circ \varphi_{ji} = id$ on Y_{ijk}^0. This translates to $\alpha_{ik}+\alpha_{kj}+\alpha_{ji}=0$. In other words, (β_i, α_{ji}) is a 2-cocycle in the "truncated Čech-De Rham complex": the Čech complex of $\Omega^{\geq 1}_{Y_0} := (0 \to \Omega^1_{Y_0} \to \Omega^2_{Y_0} \to \ldots)$. And (β_i, α_{ji}) is defined uniquely up to adding a 2-coboundary: $(d\alpha_i, \alpha_j - \alpha_i)$. So we arrive at:

Conclusion: Filtered quantizations of $\pi_* \mathcal{O}_Y$ (a.k.a. sheaves of TDO on Y_0) are classified (up to isomorphism of quantizations) by the hypercohomology $H^2(\Omega^{\geq 1}_{Y_0})$.

Rem: in a number of situations $H^2(\Omega^{\geq 1}_{Y_0})$ is the same as $H^2_{DR}(Y_0)$ $= H^2(Y_0, \mathbb{C})$. For example, this is case when Y_0 is "pure" (non-diagonal Hodge #'s vanish), which is the case when Y_0 admits a stratification by affine spaces. The parabolic flag varieties G/P have this property (take the Schubert stratification).