Lecture 19.

1) TDO, cont'd.

2) Quantum Hamiltonian reduction.

Ref: [G], Sec. 2.

1.0) Reminder

We are currently studying the questions of constructing & classifying filtered quantizations. Last time we have constructed & classified filtered quantizations of T^*Y_0, where Y_0 is a smooth variety: they are classified by $H^2(Y_0, \Omega_{Y_0}^{\geq 1})$. In fact, this answer fits the general pattern of classifying quantizations of Poisson varieties that are smooth (or, more generally, singular symplectic & terminal) w. some additional conditions.

1.1) DO's in line bundles.

Let Y_0 be a smooth variety & $\mathcal{L}$ be a line bundle. We cover $Y_0 = \bigcup_i Y_0^i$ by open affines s.t $\mathcal{L}|_{Y_0^i} \xrightarrow[\varphi_i]{\sim} \mathcal{O}_{Y_0^i}$. This gives rise to a 1-cocycle $f_{ij} \in \Gamma(Y_0^{ij}, \mathcal{O}^\times)$ via $f_{ji} := \varphi_j \varphi_i^{-1}(1)$. So we get a 1-cocycle of 1-forms $\alpha_{ji} := -f_{ji}^{-1} df_{ji}$, note that $d\alpha_{ij} = 0$. So $(\beta_i = 0, \alpha_{ji})$ is a 2-cocycle in the truncated Čech-De Rham complex (Sec 2.2 of Lec 18) and hence gives rise to a sheaf of TDO, to be denoted by $\mathcal{D}_{\mathcal{L}}$.

Proposition: $\mathcal{D}_{\mathcal{L}}$ acts on $\mathcal{L}$ extending the $\mathcal{O}_Y$-module structure.

Proof/construction: recall that $\mathcal{D}_{\mathcal{L}}$ is constructed as follows. We have $\mathcal{D}_{\mathcal{L}}(Y_0^i) \xrightarrow[\Phi_i]{\sim} \mathcal{D}(Y_0^i) \rightsquigarrow \Phi_j \Phi_i^{-1}: \mathcal{D}(Y_0^{ij}) \longrightarrow \mathcal{D}(Y_0^{ij})$, $f \mapsto f$, $\xi \mapsto \xi - f_{ji}^{-1} \xi . f_{ji}$. For open affine U, $\sigma \in \Gamma(U, \mathcal{L})$, $\delta \in \mathcal{D}_{\mathcal{L}, \leq 1}(U)$, set $\varphi_i(\delta\sigma) := \underbrace{\Phi_i(\delta)}_{DO} \underbrace{\varphi_i(\sigma)}_{function} \in \mathbb{C}[U \cap Y_0^i]$. We need to check this is well-defined: $\varphi_i^{-1}(\Phi_i(\delta)\varphi_i(\sigma)) = \varphi_j^{-1}(\Phi_j(\delta)\varphi_j(\sigma))$ [move φ_j to the l.h.s]

$\Leftrightarrow$ [$\varphi_i(\sigma) =: f \Rightarrow \varphi_j(\sigma) = f_{ji} f$, $\Phi_i(\delta) =: \xi + g \Rightarrow \Phi_j(\delta) = \xi - f_{ji}^{-1} \xi . f_{ji} + g$]

$f_{ji}(\xi . f + gf) = \xi(f_{ji} f) + (g - f_{ji}^{-1} \xi . f_{ji}) f_{ji} f$ - true equality. The $\mathcal{D}_{\mathcal{L}, \leq 1}$-action extends to $\mathcal{D}_{\mathcal{L}}$. Details are left as an exercise. (hint: it's enough to check the relations on each Y_i^0). □

Remarks: 1) As a sheaf of filtered algebras $\mathcal{D}_{\mathcal{L}} \simeq \mathcal{L} \otimes \mathcal{D} \otimes \mathcal{L}^{-1}$ (w. $\mathcal{D}_{\mathcal{L}, \leq i} = \mathcal{L} \otimes \mathcal{D}_{\leq i} \otimes \mathcal{L}^{-1}$)

2) The class $(0, \alpha_{ij}) \in H^2(\Omega_Y^{\geq 1})$ is called the 1st Chern class

of $\mathcal{L}$ and is denoted by $c_1(\mathcal{L})$ (the usual "topological Chern class" is the image of this under $H^2(\Omega_Y^{\geq 1})\longrightarrow H^2(\Omega_Y)=H^2(Y,\mathbb{C})$).
Note that $c_1: Pic(Y)\longrightarrow H^2(\Omega_Y^{\geq 1})$ is additive. For $\mathcal{L}_k\in Pic(Y)$, $k=1,\dots,s$, and $z_i\in\mathbb{C}$, it makes sense to speak about $\mathcal{D}_{\mathcal{L}_1^{z_1}\dots\mathcal{L}_s^{z_s}}$ - this is the sheaf of TDO w. parameter $\sum_{i=1}^{s} z_i c_1(\mathcal{L}_i)$ - but $\mathcal{L}_1^{z_1}\dots\mathcal{L}_s^{z_s}$ is an actual line bundle iff $z_i\in\mathbb{Z}$.

1.2) The case of $Y_0=G/P$.

In this case $H^2(\Omega_{Y_0}^{\geq 1})\xrightarrow{\sim} H^2_{DR}(Y_0)$, see the last remark in Sec. 2.2 of Lec 18: G/P admits affine paving. In more detail, assume $P=P(\Pi_0)$ for $\Pi_0\subset\Pi$ (see Sec 2.1 of Lec 13 for the notation). Let $W_0\subset W$ denote the subgroup generated by s_α w. $\alpha\in\Pi_0$. We have the parabolic Schubert decomposition:

$$G/P=\coprod_w B^- wP/P,$$

where B^- is the negative Borel and w runs over the elements in W that are of minimal length in wW_0. Each B^-wP/P is an affine space. A basis in $H^2(G/P,\mathbb{Z})$ is indexed by codim 1 components; these are exactly $B^-s_\alpha P/P$ for $\alpha\in\Pi\setminus\Pi_0$. Moreover c_1 gives an isomorphism $\mathfrak{X}(L)\simeq Pic(Y)\xrightarrow{\sim} H^2(G/P,\mathbb{Z})$: the fundamental weights ϖ_α corresponding to $\alpha\in\Pi\setminus\Pi_0$ form a basis in $\mathfrak{X}(L)$ & ϖ_α is sent to the basis element corresponding to $\overline{B s_\alpha P/P}$. Let $\mathcal{L}_\alpha$ denote the line bundle corresponding to ϖ_α.

The conclusion of this discussion is that every sheaf of TDO on G/P has the form $\mathcal{D}_{\mathcal{L}^{\underline{z}}}$ for $\underline{z}=(z_\alpha)_{\alpha\in\Pi\setminus\Pi_0}$ & $\mathcal{L}^{\underline{z}}=\bigotimes_{\alpha\in\Pi\setminus\Pi_0}\mathcal{L}_\alpha^{z_\alpha}$.

2) Quantum Hamiltonian reductions.

2.0) Motivation.

As explained in the previous lecture, our goal is to construct quantizations of induced varieties $\operatorname{Ind}_P^G(X_L)$. If $X_L=\{0\}$, then $Y=\operatorname{Ind}_P^G(X_L)=T^*(G/P)$ & the quantizations are the sheaves of TDO, parameterized by $H^2(G/P,\mathbb{C})\simeq(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. In general, $\operatorname{Ind}_P^G(X_L)$ is obtained as (classical) Hamiltonian reduction. Our goal is to construct, for each $\lambda\in(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$, a filtered quantization $\mathcal{D}_\lambda$ of Y (we'll define carefully what this means later). To construct $\mathcal{D}_\lambda$ we'll use the quantum Hamiltonian reduction that we'll start to discuss now.

2.1) Quantum comoment maps.

Let $\mathcal{A}$ be an associative unital algebra and H be an algebraic group that acts on $\mathcal{A}$ rationally by automorphisms. By differentiation, this gives rise to a Lie algebra homomorphism $\mathfrak{h} \to \operatorname{Der}(\mathcal{A})$, $\xi \mapsto \xi_{\mathcal{A}}$.

Definition: By a quantum comoment map for $H \curvearrowright \mathcal{A}$ we mean an H-equivariant linear map $\Phi: \mathfrak{h} \to \mathcal{A}$ s.t. $[\Phi(\xi), \cdot] = \xi_{\mathcal{A}}$ $\forall \xi \in \mathfrak{h}$ (note that Φ is a Lie algebra homomorphism, exercise).

Example 1: Let $\mathcal{A} = U(\mathfrak{h})$ equipped with the usual H-action. The natural inclusion $\mathfrak{h} \hookrightarrow U(\mathfrak{h})$ is a quantum comoment map.

Example 2: Let Y_0 be a smooth affine variety w. an H-action. Take $\mathcal{A} = D(Y_0)$ and equip it with the induced H-action. For $\xi \in \mathfrak{h}$, set $\Phi(\xi) = \xi_{Y_0}$ so that $\Phi: \mathfrak{h} \to D(Y_0)$ is linear & H-equivariant. So, $f \in \mathbb{C}[Y_0]$, $\eta \in \operatorname{Vect}(Y_0)$, we have $[\Phi(\xi), f] = \xi_{Y_0} \cdot f = \xi_{D(Y_0)} f$, $[\Phi(\xi), \eta] = [\xi_{Y_0}, \eta] = \xi_{D(Y_0)} \cdot \eta$. It follows that Φ is a quantum comoment map.

Now assume that $\mathcal{A}$ is equipped w. an algebra filtration, $\mathcal{A} = \bigcup_{i\geqslant 0} \mathcal{A}_{\leqslant i}$ preserved by H. Assume further that $\deg[\cdot,\cdot] \leqslant -d$ for $d \in \mathbb{Z}_{\geqslant 0}$ (i.e. $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{\leqslant i+j-d}$). Finally, assume that a quantum comoment map Φ has image in $\mathcal{A}_{\leqslant d}$. Let $A = \operatorname{gr}\mathcal{A}$.

Exercise: The action $H \curvearrowright A$ is Hamiltonian w. comoment map $\varphi := \Phi + \mathcal{A}_{\leqslant d-1} : \mathfrak{h} \to A_d$ is a classical comoment map.

Applying this to the previous two examples we recover the comoment maps from Sec 2.2 of Lec 2.

Rem: As for the classical comoment maps, if $\lambda \in (\mathfrak{h}^*)^H$ & Φ is a quantum comoment map, then $\Phi + \lambda$ is also a quantum comoment map.

2.2) Quantum Hamiltonian reduction.

Let $\mathcal{A}, H, \Phi$ be as in Sec 2.1. The left ideal $\mathcal{A}\Phi(\mathfrak{h}) \subset \mathcal{A}$ is H-stable so we can talk about $[\mathcal{A}/\mathcal{A}\Phi(\mathfrak{h})]^H \subset \mathcal{A}/\mathcal{A}\Phi(\mathfrak{h})$ Note that $a + \mathcal{A}\Phi(\mathfrak{h}) \in [\mathcal{A}/\mathcal{A}\Phi(\mathfrak{h})]^H$ is also $\mathfrak{h}$-invariant

$\Leftrightarrow \xi_{\mathcal{A}} a = [\Phi(\xi), a] \in \mathcal{A}\Phi(\mathfrak{h})$. Using this we get:

Lemma: There's a unique algebra structure on $[\mathcal{A}/\mathcal{A}\Phi(\mathfrak{h})]^H$ s.t. $(a + \mathcal{A}\Phi(\mathfrak{h}))\cdot(b + \mathcal{A}\Phi(\mathfrak{h})) = ab + \mathcal{A}\Phi(\mathfrak{h})$.

We denote the resulting algebra by $\mathcal{A}///_0 H$.

Exercise: Assume H is connected. Construct an isomorphism $\mathcal{A}///_0 H \xrightarrow{\sim} \mathrm{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{h}))^{opp}$.

Rem: Fix Φ. Then we have a family of algebras $\mathcal{A}///_\lambda H := (\mathcal{A}/\mathcal{A}\{\xi - \langle\lambda,\xi\rangle \mid \xi \in \mathfrak{h}\})^H$.

2.3) Quantization commutes w. reduction

Now assume that $\mathcal{A}$ is filtered w. $\deg[\cdot,\cdot] \leqslant -d$ & $\mathrm{im}\,\Phi \subset \mathcal{A}_{\leqslant d}$. Set $A := \mathrm{gr}\,\mathcal{A}$, $\varphi := \Phi + \mathcal{A}_{\leqslant d-1} : \mathfrak{g} \to A_d$. The quotient $\mathcal{A}/\mathcal{A}\Phi(\mathfrak{h})$ inherits a filtration from $\mathcal{A}$, and it's preserved by H, so $(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{h}))^H$ is a filtered algebra w. $\deg[\cdot,\cdot] \leqslant -d$. One can ask for conditions that guarantee that $(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{h}))^H$ is a filtered quantization of the classical Hamiltonian reduction $(A/A\varphi(\mathfrak{h}))^H$. We assume that A is finitely generated.

First, note that $A\varphi(\mathfrak{h}) \subset \operatorname{gr}[\mathcal{A}\Phi(\mathfrak{h})]$.

Lemma 1: Suppose that for a basis $\xi_1, \ldots \xi_n \in \mathfrak{h}$, the sequence $\varphi(\xi_1), \ldots, \varphi(\xi_r) \in A$ is regular. Then $A\varphi(\mathfrak{h}) = \operatorname{gr}[\mathcal{A}\Phi(\mathfrak{h})]$.

Proof: This is a special case of Claim in Sec 1 of Lec 11: condition (6) there follows b/c $\Phi(\mathfrak{h}) \subset \mathcal{A}$ is Lie subalgebra. $\square$

Second, taking H-invariants is left exact (and exact iff H is reductive) so $\operatorname{gr}[(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{h}))^H] \hookrightarrow [A/A\varphi(\mathfrak{h})]^H$ (an isomorphism when H is reductive).

We will need a special situation (for general H). Let $X := \operatorname{Spec} A$ & $\mu: X \longrightarrow \mathfrak{h}^*$ be the moment map (dual to φ).

Lemma: Suppose that H acts on $\mu^{-1}(0)$ freely & $\exists$ affine Y s.t. $\mu^{-1}(0) \to Y$ is a principal H-bundle. Then $gr[(\mathcal{A}/\mathcal{A}\varphi(\mathfrak{h}))^H] \xrightarrow{\sim} [A/A\varphi(\mathfrak{h})]^H = \mathbb{C}[Y]$.

Sketch of proof: Since $H \curvearrowright \mu^{-1}(0)$ freely, we have that the vector fields $\xi_{1,x}, \dots \xi_{n,x}$ are linearly independent at all points of $x \in \mu^{-1}(0) \Leftrightarrow d_x\varphi(\xi_1), \dots d_x\varphi(\xi_n)$ are linearly independent $\forall x \in \mu^{-1}(0)$, in particular, $\varphi(\xi_1), \dots, \varphi(\xi_n)$ form a regular sequence.

Hence for $\mathcal{B} = \mathcal{A}/\mathcal{A}\varphi(\mathfrak{h})$ & $B = A/A\varphi(\mathfrak{h})$ we have $gr\,\mathcal{B} \xrightarrow{\sim} B$. Now we need to show that $gr(\mathcal{B}^H) \xrightarrow{\sim} B^H \Leftrightarrow$ $\forall i$ we have SES $0 \to \mathcal{B}^H_{\leq i-1} \to \mathcal{B}^H_{\leq i} \to B_i^H \to 0$.

This will follow if we check that B is injective in the category of rational H-representations (then B_i is injective as a direct summand of an injective representation). We won't give a proof here – we'll leave it for a separate note. $\square$