MATH 720, Lecture 2.

1) Poisson manifolds & Classical Mechanics.
2) Hamiltonian actions & moment maps.

Refs: [CdS], Secs 1,2,18,21,22; [CG], 1.1, 1.2, 1.4.

In this lecture we continue to review symplectic & Poisson manifolds. We also touch on the symmetry in Classical Mechanics: Hamiltonian actions of Lie groups.

1.1) Examples of Poisson manifolds.

1) Let V be a vector space and $P\in \Lambda^2 V (\subset \Gamma(\Lambda^2 T_V))$. The bracket $\{\cdot,\cdot\}_P$ on $C^\infty(V)$ can be described as follows. Pick a basis $x_1,\dots x_n \in V^*$ and let $p_{ij}:=P(x_i,x_j)$. Then $\{f,g\}_P=\sum_{i,j=1}^n p_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$. The bracket is Poisson: it's enough to check the Jacobi id'y on $x_1,\dots x_n$ b/c they functionally generate $C^\infty(V)$, and there it's straightforward. The Poisson bivector is non-degenerate $\Leftrightarrow$ P is, i.e. V is a symplectic vector space.

2) Let M_0 be a manifold. Set $M := T^*M_0$. This is a symplectic manifold. We won't need a formula for the symplectic form, see [CdS], Sec 2, [CG], Sec 1.1. But we'll need the formulas for brackets of some elements of $C^\infty(M)$. Note that $C^\infty(M_0)$ (resp., $\mathrm{Vect}(M_0) := \Gamma(T_{M_0})$) embed into $C^\infty(M)$ as the subsets of functions that are constant (resp. linear) on fibers of $T^*M_0 \twoheadrightarrow M_0$. Then we have:

$$\{f_1, f_2\} = 0, \quad \{\xi_1, f_1\} = \xi_1 \cdot f_1, \quad \{\xi_1, \xi_2\} = [\xi_1, \xi_2] \qquad (1)$$
$$f_1, f_2 \in C^\infty(M_0), \ \xi_1, \xi_2 \in \mathrm{Vect}(M_0).$$

Note that (1) uniquely determines the bracket. Note also that if M_0 is a vector space, then the symplectic form on $M = M_0 \oplus M_0^*$ arises in the previous example.

3) Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $\mathfrak{g}^*$ carries the unique Poisson structure s.t. for $\xi, \eta \in \mathfrak{g}$ (viewed as linear functions on $\mathfrak{g}^*$) we have $\{\xi, \eta\} = [\xi, \eta]$. The Poisson bivector P can be described as follows: for $\alpha \in \mathfrak{g}^*$ $\xi, \eta \in T_\alpha \mathfrak{g}^* = \mathfrak{g}$ have $\langle P_\alpha, \xi \wedge \eta \rangle = \langle \alpha, [\xi, \eta] \rangle$. To check the

details is an exercise.

Note that this Poisson structure is degenerate: $P_0 = 0$.

Rem: We have worked w. real manifolds & C^∞-functions. But the definitions & constructions carry over to the complex analytic & algebraic settings essentially verbatim. For example, let X_0 be a smooth and, for simplicity, affine variety over $\mathbb{C}$. Then its tangent bundle $X = T^*X_0$ is defined as $\operatorname{Spec}(S_{\mathbb{C}[X_0]}(\operatorname{Vect} X_0))$ and (1) extends to a Poisson bracket on $S_{\mathbb{C}[X_0]}(\operatorname{Vect}(X_0))$. If X_0 is not affine, we define the Poisson structure on $T^*X_0^i$ for open affine cover $X_0 = \bigcup_i X_0^i$. These Poisson structures glue to a Poisson structure on $\mathcal{O}_{T^*X_0}$. The variety T^*X_0 is symplectic.

1.2) Classical mechanics

Space of states: a Poisson manifold/variety M

Observables: Functions (C^∞/analytic/algebraic) on M

Hamiltonian: One of the observables, H.

Evolution equation: Hamilton equation $\dot{f}(t) = \{H, f(t)\}$.

And the right notion of symmetry will be explained in the next section.

2) Hamiltonian actions & moment maps.

2.1) Definitions.

Let G be a Lie group acting on a manifold M. This gives rise to a G-equivariant Lie algebra homomorphism $\xi \mapsto \xi_M : \mathfrak{g} \to \mathrm{Vect}(M)$ (if $\gamma(t)$ is a curve in G w. $\gamma(0)=1$, $\frac{d\gamma}{dt}\big|_{t=0}=\xi$, then, for $m\in M$, $\xi_{M,m}$ is the tangent vector to $\gamma(t)m$).

Now assume M is Poisson and G preserves $\{\cdot,\cdot\}$. For $f\in C^\infty(M)$, $\{f,\cdot\}: C^\infty(M)\to C^\infty(M)$ is a derivation, we write $v(f)$ when we view it as a vector field. The map $f\mapsto v(f)$ is also a G-equivariant Lie algebra homomorphism.

Definition: • A (classical) comoment map is a G-equivariant linear map $\xi \mapsto H_\xi : \mathfrak{g}\to C^\infty(M)$ s.t. the following diagram commutes

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\xi\mapsto\xi_M"] \arrow[dr, dashed, "\xi\mapsto H_\xi"'] & \mathrm{Vect}(M) \\
 & C^\infty(M) \arrow[u, "f\mapsto v(f)"']
\end{tikzcd}$$

Equivalently, $\xi_M = v(H_\xi)$ $(=\langle P, dH_\xi\rangle)$ $\forall\, \xi\in\mathfrak{g}$.

Exercise: $\xi \mapsto H_\xi$ is a Lie algebra homomorphism.

• By a Hamiltonian G-action we mean a G-action on M preserving $\{\cdot,\cdot\}$ together w. a classical comoment map.

• The moment map μ for a Hamiltonian G-action on M is the map $\mu: M \longrightarrow \mathfrak{g}^*$ given by

$$\langle \mu(m), \xi \rangle := H_\xi(m), \quad m \in M, \ \xi \in \mathfrak{g}.$$

• The symmetry of a classical Hamiltonian system on M with Hamiltonian H is a Hamiltonian G-action on M preserving H.

2.2) Examples

The verification of the conditions is left as an exercise (or look at the refs).

1) Consider the coadjoint action of G on $\mathfrak{g}^*$. It's Hamiltonian w. comoment map sending $\xi \in \mathfrak{g}$ to ξ viewed as a linear function on $\mathfrak{g}^*$. The moment map is the identity.

2) Let G act on a manifold M_0. The induced action on $M = T^*M_0$ is Hamiltonian w. $H_\xi := \xi_{M_0}$.

3) Let V be a symplectic vector space w. form ω and G be a Lie group acting on V via a homomorphism $G \longrightarrow Sp(V)$ (so that $\xi_V \in \mathfrak{sp}(V)$). This action is Hamiltonian w. comoment map $H_\xi(v) = \frac{1}{2}\omega(\xi v, v)$ (this exercise is on computing differentials of quadratic functions).

Remark: Let's explain the relevance of the comoment map for Classical Mechanics. Suppose $H \in C^\infty(M)^G$ (G-invariant). Then $\{H_\xi, H\} = \xi_M H = 0$. The Hamilton equation gives

$$\dot{H}_\xi(t) = \{H, H_\xi\} = 0$$

so H_ξ is constant on the trajectories of the system and hence is a conserved quantity (this is the Noether principle: conserved quantities correspond to continuous symmetries).

2.3) Coadjoint orbits as symplectic manifolds.

Let G be a connected Lie group, $\mathfrak{g} = Lie(G)$. We consider the coadjoint representation $G \curvearrowright \mathfrak{g}^*$ and hence can talk about coadjoint orbits. Turns out they are symplectic manifolds that, essentially exhaust all transitive Hamiltonian actions.

In this section we will equip any coadjoint G-orbit $G\alpha$ ($\alpha\in\mathfrak{g}^*$) w. a G-invariant symplectic form s.t. the embedding $G\alpha\hookrightarrow\mathfrak{g}^*$ is a moment map.

Recall (Sec 2.2 of Lec 2) that for the Poisson bivector $\mathcal{P}$ on $\mathfrak{g}^*$ we have $\langle\mathcal{P}_\beta, \xi\wedge\eta\rangle = \langle\beta,[\xi,\eta]\rangle$. Note that $G\alpha\subset\mathfrak{g}^*$ is an immersed submanifold & $T(G\beta) = \mathfrak{g}.\beta\subset\mathfrak{g}^*$. under the coadjoint rep'n

Set $\mathfrak{g}_\beta := \{\xi\in\mathfrak{g} \mid \langle\beta,[\xi,\eta]\rangle = 0\ \forall\eta\in\mathfrak{g}\} = \ker[\mathfrak{g}\twoheadrightarrow\mathfrak{g}_\beta, \xi\mapsto\xi.\beta]$.

The following important exercise gives a construction of a symplectic form on $G\alpha$ known as the Kirillov-Kostant form to be denoted by ω_{KK}.

Exercise: (i) $\langle\mathcal{P}_\beta,\xi\wedge\eta\rangle = 0\ \forall\eta\in\mathfrak{g}_\beta \Leftrightarrow \xi\in\mathfrak{g}_\beta$. Deduce that $\mathcal{P}$ descends to a symplectic form, $\omega_{KK,\beta}$ on $\mathfrak{g}/\mathfrak{g}_\beta\simeq T_\beta(G\beta)$

(ii) Justify that the form ω_{KK} on $G\alpha$ whose value at $\beta\in G\alpha$ is $\omega_{KK,\beta}$ is C^∞. Show that $\omega_{KK,\beta}$ is G-invariant.

(iii) The form ω_{KK} is closed - and hence symplectic (hint: ω_{KK} satisfies $\xi_M\cdot\omega_{KK} = 0$ (G-invariance) & $\omega_{KK,\beta}(\xi_{M,\beta},\eta_{M,\beta}) = \langle\beta,[\xi,\eta]\rangle$. One can then check ω_{KK} is closed by using the "Cartan magic formula").

(iv) The inclusion $G\alpha \hookrightarrow \mathfrak{g}^*$ is a moment map for $G \curvearrowright G\alpha$.

Example*: $G = U(n)$ (so $\mathfrak{g} \simeq \mathfrak{g}^*$ via $(x,y) = -tr(xy)$), $\alpha = diag(i,0,..0)$
$\Rightarrow G\alpha \simeq \mathbb{CP}^{n-1}$ & ω_{KK} is the Fubini-Study form. More generally, we can equip generalized (a.k.a. parabolic) flag varieties for complex s/simple Lie groups, $G_{\mathbb{C}}$, w. (real) symplectic forms that are parts of Kähler structures (take the compact form G of $G_{\mathbb{C}}$ & appropriate $\alpha \in \mathfrak{g}^*$).