MATH 720, Lecture 2.

1) Poisson manifolds & Classical Mechanics.
2) Hamiltonian actions & moment maps.

Refs: [CdS], Secs 12,18,21,22; [CG], 1.1,12,14.

In this lecture we continue to review symplectic & Poisson manifolds. We also touch on the symmetry in Classical Mechanics: Hamiltonian actions of Lie groups.

1.1) Examples of Poisson manifolds.

1) Let $V$ be a vector space and $P \in \Lambda^2 V (= \Gamma(\Lambda^2 T_V))$. The bracket $\{ ; \}_P$ on $C^\infty(V)$ can be described as follows.

Pick a basis $x_1, \ldots, x_n \in V^*$ and let $p_{ij} = P(x_i, x_j)$. Then

$$\{f, g\}_P = \sum_{i,j=1}^n p_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$ The bracket is Poisson: it's enough to check the Jacobi identity on $x_1, \ldots, x_n$ since they functionally generate $C^\infty(V)$, and there it's straightforward. The Poisson bivector is non-degenerate $\Leftrightarrow P$ is, i.e. $V$ is a symplectic vector space.
2) Let $M_0$ be a manifold. Set $M := T^*M_0$. This is a symplectic manifold. We won’t need a formula for the symplectic form, see [CdS], Sec 2, [CG], Sec 1.1 But we’ll need the formulas for brackets of some elements of $C^\infty(M)$. Note that $C^\infty(M_0)$ (resp., $\text{Vect}(M_0) := \Gamma(TM_0)$) embed into $C^\infty(M)$ as the subsets of functions that are constant (resp. linear) on fibers of $T^*M_0 \to M_0$. Then we have:

\[ \{f, g\} = 0, \quad \{\xi_1, \xi_2\} = \xi_1 \cdot h, \quad \{\xi_1, \xi_2\} = [\xi_1, \xi_2] \quad (1) \]

\[ f, g \in C^\infty(M_0), \quad \xi_1, \xi_2 \in \text{Vect}(M_0). \]

Note that (1) uniquely determines the bracket. Note also that if $M_0$ is a vector space, then the symplectic form on $M = M_0 \oplus M_0^*$ arises in the previous example.

3) Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $\mathfrak{g}^*$ carries the unique Poisson structure s.t. for $x, y \in \mathfrak{g}$ (viewed as linear functions on $\mathfrak{g}^*$) we have $\{x, y\} = [[x, y]]$. The Poisson bivector $P$ can be described as follows: for $x, y \in T_x^*\mathfrak{g} \cong \mathfrak{g}$ have $\langle P, (x^\flat, y^\flat) \rangle = \langle \alpha, [x, y] \rangle$. To check the
Note that this Poisson structure is degenerate: $P_0 = 0$.

**Rem:** We have worked w. real manifolds & $C^\infty$-functions. But the definitions & constructions carry over to the complex analytic & algebraic settings essentially verbatim. For example, let $X_0$ be a smooth and, for simplicity, affine variety over $\mathbb{C}$. Then its tangent bundle $X = T^* X_0$ is defined as $\text{Spec} (S_{\mathbb{C}[X_0]} (\text{Vect } X_0))$ and (1) extends to a Poisson bracket on $S_{\mathbb{C}[X_0]} (\text{Vect}(X_0))$. If $X_0$ is not affine, we define the Poisson structure on $T^* X_0^i$ for open affine cover $X_0 = U X_0^i$. These Poisson structures glue to a Poisson structure on $\mathcal{O}_{T^* X_0}$. The variety $T^* X_0$ is symplectic.

1.2) **Classical mechanics**

**Space of states:** a Poisson manifold/varietiy $M$

**Observables:** Functions ($C^\infty$/analytic/algebraic) on $M$

**Hamiltonian:** One of the observables, $H$.

**Evolution equation:** Hamilton equation $\dot{f}(t) = \{H, f(t)\}$. 

Details is an exercise.
And the right notion of symmetry will be explained in the next section.

2) Hamiltonian actions & moment maps.

2.1) Definitions.

Let $G$ be a Lie group acting on a manifold $M$. This gives rise to a $G$-equivariant Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}_M : g \mapsto \text{Vect}(M)$ (if $\gamma(t)$ is a curve in $G$ w/ $\gamma(0) = e$, $\frac{d}{dt} \gamma(0) = \mathfrak{g}$, then, for $m \in M$, $\mathfrak{g}_M$ is the tangent vector to $\gamma(t)m$).

Now assume $M$ is Poisson and $G$ preserves $\mathfrak{g}$. For $f \in C^\infty(M)$, $\mathfrak{g} \cdot f : C^\infty(M) \to C^\infty(M)$ is a derivation, we write $\mathfrak{g}\mathfrak{g}(f)$ when we view it as a vector field. The map $f \mapsto \mathfrak{g}\mathfrak{g}(f)$ is also a $G$-equivariant Lie algebra homomorphism.

**Definition:** A (classical) comoment map is a $G$-equivariant linear map $\mathfrak{g} \mapsto H_\mathfrak{g} : \mathfrak{g} \to C^\infty(M)$ s.t. the following diagram commutes

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\mathfrak{g} \mapsto \mathfrak{g}_M} & \text{Vect}(M) \\
\uparrow H_\mathfrak{g} \quad & & \quad \uparrow f \mapsto \mathfrak{g}\mathfrak{g}(f) \\
\mathfrak{g} & \quad \xrightarrow{\mathfrak{g} \mapsto H_\mathfrak{g}} & \quad C^\infty(M)
\end{array}
\]

Equivalently, $\mathfrak{g}_M = \mathfrak{g}(H_\mathfrak{g}) = \langle \mathfrak{p}, \text{d}H_\mathfrak{g} \rangle \forall \mathfrak{g} \in \mathfrak{g}$. 

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Exercise: \( \xi \mapsto H_\xi \) is a Lie algebra homomorphism.

- By a Hamiltonian \( G \)-action we mean a \( G \)-action on \( M \) preserving \( \{ ; \} \) together w. a classical comoment map.
- The moment map \( \mu \) for a Hamiltonian \( G \)-action on \( M \) is the map \( \mu: M \to \mathfrak{g}^* \) given by
  \[
  \langle \mu(m), \xi \rangle := H_\xi(m), \ m \in M, \ \xi \in \mathfrak{g}.
  \]
- The symmetry of a classical Hamiltonian system on \( M \) with Hamiltonian \( H \) is a Hamiltonian \( G \)-action on \( M \) preserving \( H \).

2.2) Examples

The verification of the conditions is left as an exercise (or look at the refs).

1) Consider the coadjoint action of \( G \) on \( \mathfrak{g}^* \). It’s Hamiltonian w. comoment map sending \( \xi \in \mathfrak{g} \) to \( \xi \) viewed as a linear function on \( \mathfrak{g}^* \). The moment map is the identity.

2) Let \( G \) act on a manifold \( M_0 \). The induced action on \( M = T^*M_0 \) is Hamiltonian w. \( H_\xi := \xi_{|M_0} \).
3) Let $V$ be a symplectic vector space with form $\omega$ and $G$ be a Lie group acting on $V$ via a homomorphism $C \rightarrow Sp(V)$ (so that $g \in Sp(V)$). This action is Hamiltonian with comoment map $H_g(v) = \frac{1}{2} \omega(gv,v)$ (this exercise is on computing differentials of quadratic functions).

**Remark:** Let's explain the relevance of the comoment map for Classical Mechanics. Suppose $H \in C^\infty(M)^G$ ($G$-invariant). Then $\{H_g, H^2\} = \delta_H H = 0$. The Hamilton equation gives

$$\dot{H}_g(t) = \{H, H_g^2\} = 0$$

so $H_g$ is constant on the trajectories of the system and hence is a conserved quantity (this is the Noether principle: conserved quantities correspond to continuous symmetries).

2.3) Coadjoint orbits as symplectic manifolds.

Let $G$ be a connected Lie group, $\mathfrak{g} = \text{Lie}(G)$. We consider the coadjoint representation $G \rightarrow \text{adj}(\mathfrak{g}^*)$ and hence can talk about coadjoint orbits. Turns out they are symplectic manifolds that essentially exhaust all transitive Hamiltonian actions.
In this section we will equip any coadjoint $G$-orbit $G_{\alpha}$ (where $G_{\alpha}$ is an immersed submanifold & $T(G_{\alpha}) = g_{\ast}$ under the coadjoint rep) with a $G$-invariant symplectic form $\omega$ such that the embedding $G_{\alpha} \hookrightarrow g^{\ast}$ is a moment map.

Recall (Sec 2.2 of Lee 2) that for the Poisson bivector $\mathcal{P}$ on $g^{\ast}$ we have $\left< \mathcal{P}_\beta, \xi \wedge \eta \right> = \left< \beta, [\xi, \eta] \right>$. Note that $G_{\alpha} \subset g^{\ast}$ is an immersed submanifold of $\mathcal{P}$.

Set $g_{\beta} = \{ \xi \in g^{\ast} \mid \beta, [\xi, \eta] = 0 \forall \eta \in g^{\ast} \} = \ker[\mathcal{P} \rightarrow g_{\beta}, \xi \mapsto \xi \beta]$. The following important exercise gives a construction of a symplectic form on $G_{\alpha}$ known as the Kirillov-Kostant form to be denoted by $\omega_{\text{KK}}$.

Exercise: (i) $\left< \mathcal{P}_\beta, \xi \wedge \eta \right> = 0 \forall \eta \in g_{\beta} \iff \xi \in g_{\beta}$. Deduce that $\mathcal{P}$ descends to a symplectic form $\omega_{\text{KK}}$ on $g_{\beta} / g_{\beta} \cong T_{\beta}(G_{\alpha})$.

(ii) Justify that the form $\omega_{\text{KK}}$ on $G_{\alpha}$ whose value at $\beta \in G_{\alpha}$ is $\omega_{\text{KK}}$ is $C^{\infty}$. Show that $\omega_{\text{KK}}$ is $G$-invariant.

(iii) The form $\omega_{\text{KK}}$ is closed and hence symplectic (hint: $\omega_{\text{KK}}$ satisfies $\xi_{M} \cdot \omega_{\text{KK}} = 0$ (G-invariance) & $\omega_{\text{KK}, \beta} (\xi_{M} \beta, \xi_{M} \beta) = \left< \beta, [\xi, \xi] \right>$. One can then check $\omega_{\text{KK}}$ is closed by using the "Cartan magic formula").
(iv) The inclusion $G_{\mathbb{C}} \hookrightarrow \mathfrak{g}^*$ is a moment map for $G_{\mathbb{C}} \times \mathfrak{g}$.

Example: $G = U(n)$ (so $\mathfrak{g} \cong \mathfrak{g}^*$ via $(x,y) = -\text{tr}(xy)$), $d = \text{diag}(i,0,0) \Rightarrow G_{\mathbb{C}} \cong \mathbb{C}P^{n-1}$ & $w_{\text{FS}}$ is the Fubini-Study form. More generally, we can equip generalized (a.k.a. parabolic) flag varieties for complex s/simple Lie groups, $G_{\mathbb{C}}$, w. (real) symplectic forms that are parts of Kähler structures (take the compact form $G$ of $G_{\mathbb{C}}$ & appropriate $d \in \mathfrak{g}^*$).