Lecture 20.

1) TDO's vs Quantum Hamiltonian reduction.
2) Quantizations of induced varieties.

1.0) Introduction: In this section we'll give an example of computation of Quantum Hamiltonian reduction. Consider the following situation. Let Y_0 be a smooth variety, H be an algebraic group and $\tilde{Y}_0$ is a principal H-bundle on Y_0. Then H acts on $T^*\tilde{Y}_0$ and also on the sheaf $\mathcal{D}_{\tilde{Y}_0}$. Both actions are Hamiltonian w. classical/quantum comoment map $\xi \mapsto \xi_{\tilde{Y}_0}: \mathfrak{h} \to \mathrm{Vect}(\tilde{Y}_0)$ (note that we haven't discussed quantum Hamiltonian actions on sheaves. The claim of Example 2 in Sec 2.1 of Lec 19 is still true. In our case it reduces to the affine case b/c the morphism $\tilde{Y}_0 \to Y_0$ is affine (so $U \subset Y_0$ affine $\Rightarrow$ so is $\pi^{-1}(U)$).

The following proposition generalizes Example 1 in Sec 1.2 of Lec 13 (that deals w. $Y_0 = G/H$, $\tilde{Y}_0 = G$). We'll comment on a proof later.

Proposition: There's a natural symplectomorphism $T^*\tilde{Y}_0 /\!/\!/_0 H \xrightarrow{\sim} T^* Y_0$. Moreover, $\mu^{-1}(0) \to T^* Y_0$ is a principal H-bundle.

Define a q-coh't sheaf $D_{\tilde{Y}_0} /\!/\!/_\lambda H$ on Y_0 as follows: for $U \subset Y_0$ open affine set $D_{\tilde{Y}_0} /\!/\!/_\lambda H(U) := D(\pi^{-1}(U)) /\!/\!/_\lambda H$ & for inclusion $V \subset U$, the restriction map $D_{\tilde{Y}_0} /\!/\!/_\lambda H(U) \to D_{\tilde{Y}_0} /\!/\!/_\lambda H(V)$ is induced by the restriction map $D_{\tilde{Y}_0}(\pi^{-1}(U)) \to D_{\tilde{Y}_0}(\pi^{-1}(V))$.

Lemma in Sec 2.3 of Lec 19 applies and so $\forall\, \lambda \in (\mathfrak{h}^*)^H$, $D_{\tilde{Y}_0} /\!/\!/_\lambda H$ is a filtered quantization of $T^*\tilde{Y}_0 /\!/\!/_0 H = T^* Y_0$, hence a sheaf of TDO. We want to identify this sheaf of TDO for $\lambda \in \mathfrak{X}(H) \otimes_{\mathbb{Z}} \mathbb{C}$ ($\hookrightarrow \mathfrak{h}^{*H}$ via $d_1 : \mathfrak{X}(H) \to (\mathfrak{h}^*)^H$). Write $\lambda = z_1 \chi_1 + \ldots + z_k \chi_k$. Then χ_ℓ gives a line bundle on Y_0, $\mathcal{L}_{\chi_\ell}$, whose total space is $(\tilde{Y}_0 \times \underbrace{\mathbb{C}_{\chi_\ell}}_{H \text{ acts via } \chi_\ell})/H$. Set $D^\lambda_{Y_0} := D_{\bigotimes_{\ell=1}^{k} \mathcal{L}_{\chi_\ell}^{\otimes z_\ell}}$.

Theorem: We have an isomorphism of filtered quantizations

$$D_{\tilde{Y}_0} /\!/\!/_\lambda H \xrightarrow{\sim} D^\lambda_{Y_0}.$$

We won't need the theorem, so we'll only prove it in the special case $\lambda = 0$.

1.1) Vector fields on Y_0 vs $\tilde{Y}_0$.

Let $\pi: \tilde{Y}_0 \to Y_0$ be the natural morphism. Then $\mathcal{O}_{Y_0} \xrightarrow{\sim} (\pi_* \mathcal{O}_{\tilde{Y}_0})^H$.
We write $V_{Y_0}, V_{\tilde{Y}_0}$ for the sheaves of vector fields. We want to describe the former in terms of the latter, similarly to the description of functions above. Set $\mathfrak{h}_{\tilde{Y}_0} := \{\xi_{\tilde{Y}_0} \mid \xi \in \mathfrak{h}\} \subset V(\tilde{Y}_0) \rightsquigarrow$ subsheaf $\mathcal{O}_{\tilde{Y}_0} \mathfrak{h}_{\tilde{Y}_0} \subset V_{\tilde{Y}_0}$, these are exactly the vector fields tangent to the fibers of π. Note that for $U \subset Y_0$ (open affine), elements from $\mathcal{O}_{\tilde{Y}_0} \mathfrak{h}_{\tilde{Y}_0}(\pi^{-1}(U))$ annihilate $\mathbb{C}[\pi^{-1}(U)]^H = \mathbb{C}[U]$ (b/c $\xi_{\tilde{Y}_0}$ do).
So, for $\zeta \in V_{\tilde{Y}_0}(\pi^{-1}(U))/\mathcal{O}_{\tilde{Y}_0}\mathfrak{h}_{\tilde{Y}_0}(\pi^{-1}(U))$, the differentiation $\zeta: \mathbb{C}[U] \to \mathbb{C}[\pi^{-1}(U)]$ is well-defined. And H naturally acts on the quotient. For H-invariant ζ, we have $\zeta(\mathbb{C}[U]) \subset \mathbb{C}[U]$. And $\zeta: \mathbb{C}[U] \to \mathbb{C}[U]$ is a derivation (exercise).
This gives rise to an $\mathcal{O}_{Y_0}$-linear map

$$R_{Y_0}: \pi_*(V_{\tilde{Y}_0} / \mathcal{O}_{\tilde{Y}_0} \mathfrak{h}_{\tilde{Y}_0})^H \longrightarrow V_{Y_0}, \quad \zeta \mapsto \zeta|_{\mathcal{O}_{Y_0}} \qquad (1)$$

Lemma: (1) is an isomorphism.

Sketch of proof: This is easy when $\tilde{Y}_0 \xrightarrow{\sim} H \times Y_0$: here $\pi_* V_{\tilde{Y}_0} \simeq V_{Y_0} \otimes_{\mathbb{C}} \mathbb{C}[H] \oplus \mathcal{O}_{Y_0} \otimes_{\mathbb{C}} V_H(H)$. In general, it's enough to check

(1) on an open (Zariski/etale) cover, where π trivializes. The case when $\tilde{Y}_0 \to Y_0$ is Zariski locally trivial is easy, this is the case in our applications, where $\tilde{Y}_0 \to Y$ is $G \to G/P$ or $G \to G/U$. In general, let $\varphi: Z \to Y_0$ be an etale cover s.t. $Z \times_{Y_0} \tilde{Y}_0 \xrightarrow{\sim} Z \times H$. Then one checks that:

$$V_Z \xrightarrow{\sim} \varphi^* V_{Y_0},\ (V_{Z\times H}/\mathcal{O}_{Z\times H}\mathfrak{h}_{Z\times H})^H \xrightarrow{\sim} \varphi^*(V_{\tilde{Y}_0}/\mathcal{O}_{\tilde{Y}_0}\mathfrak{h}_{\tilde{Y}_0})^H \ \&$$

$\kappa_Z = \varphi^* \kappa_{Y_0}$. Then we use that φ^* is faithful & exact so if κ_Z is an isomorphism, then so is κ_{Y_0}. Details are exercise. $\square$

Now note that the Lie bracket on V_{Y_0} gives rise to a well-defined bracket on the l.h.s. of (1). Then (1) is an isom'm of Lie algebras. Details are also left as an exercise.

1.2) Theorem for $\lambda = 0$

We now prove the theorem in the case when $\lambda = 0$. Our goal is to construct a $\mathcal{O}_{Y_0}$-linear sheaf of algebras homomorphism $\mathcal{D}_{Y_0} \to \mathcal{D}_{\tilde{Y}_0} /\!/\!/ H$, which turns out to be an isomorphism.

Recall that $\mathcal{D}_{Y_0}$ is generated by $\mathcal{O}_{Y_0}, V_{Y_0}$. So we'll first construct $\mathcal{O}_{Y_0}$-linear maps $\mathcal{O}_{Y_0}, V_{Y_0} \to \mathcal{D}_{\tilde{Y}_0} /\!/\!/ H$.

Consider open affine $U \subset Y_0$. Consider the composition
$\mathbb{C}[U] \hookrightarrow \mathbb{C}[\pi^{-1}(U)] \hookrightarrow \mathcal{D}(\pi^{-1}(U)) \longrightarrow \mathcal{D}(\pi^{-1}(U))/\mathcal{D}(\pi^{-1}(U))\mathfrak{h}_{\tilde{Y}_0}$.
It's H-equivariant so the image is in $\mathcal{D}///_0 H(U)$. This gives rise to the desired map $\mathcal{O}_{Y_0} \longrightarrow \mathcal{D}_{\tilde{Y}_0} ///_0 H$.
Similarly, we have the map $V_{\tilde{Y}_0}(\pi^{-1}(U)) \longrightarrow \mathcal{D}(\pi^{-1}(U))/\mathcal{D}(\pi^{-1}(U))\mathfrak{h}_{\tilde{Y}_0}$.
It vanishes on $\mathbb{C}[\pi^{-1}(U)]\mathfrak{h}_{\tilde{Y}_0}$ and is H-equivariant. Using (1), we get the desired map $V_{Y_0} \rightarrow \mathcal{D}_{\tilde{Y}_0} ///_0 H$.

Exercise: This is a Lie algebra homomorphism.

Now that we've constructed maps $\mathcal{O}_{Y_0}, V_{Y_0} \xrightarrow{\iota} \mathcal{D}_{\tilde{Y}_0} ///_0 H$ we need to check that they satisfy the relations of $\mathcal{D}_{Y_0}$.
That the sections of $\mathcal{O}_{Y_0}$ multiply as in $\mathcal{O}_{Y_0}$ is straightforward.
That $[\iota(\xi), \iota(f)] = \iota(\xi . f)$ follows from the construction of (1).
And $[\iota(\xi), \iota(\eta)] = \iota([\xi, \eta])$ follows from the previous exercise.
So we get an $\mathcal{O}_{Y_0}$-linear algebra homomorphism

$$\mathcal{D}_{Y_0} \longrightarrow \mathcal{D}_{\tilde{Y}_0} ///_0 H. \qquad (2)$$

A check that it's an isomorphism we follow the same idea as in the proof of Lemma in the previous section.

• Case 1: $\tilde{Y}_0 = Y_0 \times H$. Then $\pi_* \mathcal{D}_{\tilde{Y}_0} \simeq \mathcal{D}_{Y_0} \otimes_{\mathbb{C}} \mathcal{D}(H)$,
$\pi_* (\mathcal{D}_{\tilde{Y}_0} / \mathcal{D}_{\tilde{Y}_0} \mathfrak{h}_{\tilde{Y}_0}) \simeq \mathcal{D}_{Y_0} \otimes \mathcal{D}(H)/\mathcal{D}(H)\mathfrak{h}_H$ &
$\mathcal{D}_{\tilde{Y}_0} /\!/\!/_0 H \simeq \mathcal{D}_{Y_0} \otimes_{\mathbb{C}} \underbrace{\mathcal{D}(H)/\!/\!/_0 H}_{\text{quant'n of } pt = \mathbb{C}} \simeq \mathcal{D}_{Y_0}$, w. (2) being the inverse of the resulting identification.

• Case 2: general – we argue as in the proof of Lemma in Sec 1.1.

Rem: Proposition in Sec 1.0 can be proved along the same lines.

2) Quantizations of induced varieties.

Recall that the induced variety $\mathrm{Ind}_P^G(X_L)$ (w. $X_L = \mathrm{Spec}\, \mathbb{C}[\tilde{\mathbb{O}}_L]$) is obtained as follows. Let $P = L \ltimes U$ be a Levi decomposition. We consider the action of L on G/U: $\ell.(gU) = (g\ell^{-1}U)$, then induced action on $T^*(G/U)$ and the diagonal action on $T^*(G/U) \times X_L$. It's Hamiltonian w. moment map $\mu_L : ([g,\alpha], x) \mapsto -\alpha|_{\mathfrak{l}} + \mu(x)$, where $\mu: X_L \to \mathfrak{l}^*$ is the moment map. Then $Y = \mathrm{Ind}_P^G(X_L) := \mu_L^{-1}(0)/L = G \times^P \{(\alpha, x) \mid \alpha|_{\mathfrak{l}} = \mu(x)\}$.

Note that we have the projection $Y \xrightarrow{\pi} G/P$ and the fiberwise $\mathbb{C}^\times$-action: $t.[g,(\alpha,x)] = [g,(t^2\alpha, t.x)]$. The morphism π is affine $\rightsquigarrow$ we can replace Y w. the sheaf $\pi_* \mathcal{O}_Y$ of graded Poisson algebras w. $\deg\{\cdot,\cdot\} = -2$. And we can talk about its filtered quantizations just as in Sec 2 of Lec 18 in the case of T^*Y_0. Our goal is: for $\lambda \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ construct a quantization $\mathcal{D}_\lambda$ of $\pi_* \mathcal{O}_Y$.

We'll use quantum Hamiltonian reduction. For this, we need to start w. a quantization of $T^*(G/U) \times X_L$ to be reduced.

The first factor is quantized by $\mathcal{D}_{G/U}$. Assume from now on that X_L is $\mathbb{Q}$-factorial & terminal. Here's a fact whose proof we may give at some point later.

Fact: $\mathbb{C}[X_L]$ has a unique filtered quantization. Denote it by $\mathcal{A}_L$

So consider $\mathcal{D}_{G/U} \otimes \mathcal{A}_L$ viewed as a sheaf of filtered algebras on G/U. We are going to equip it w. a Hamiltonian L-action lifting that on $\pi_* \mathcal{O}_Y$. The group L acts on $\mathcal{D}_{G/U}$ w. quantum comoment map $\xi \mapsto \xi_{G/U}$. Now we need to establish a Hamiltonian action of L on $\mathcal{A}_L$. It's existence follows from:

Proposition: Let A be a positively graded Poisson algebra w. $\deg\{\cdot,\cdot\}=-d$. Let H be a s/simple group w. Hamiltonian action on A & comoment map $\varphi:\mathfrak{h}\to A_d$. Let $\mathcal{A}$ be a filtered quantization of A. Then $L\curvearrowright\mathcal{A}$ lifts to a Hamiltonian action w. quantum comoment map $\Phi:\mathfrak{h}\to\mathcal{A}_{\leq d}$ w. $\Phi+\mathcal{A}_{\leq d-1}=\varphi$.

Proof: Essentially repeats its classical counterpart, Sec 1.1 of Lec 17. We can replace H w. a quotient to assume that $\varphi:\mathfrak{h}\hookrightarrow A_d$. Then consider $\tilde{\mathfrak{h}}:=$preimage of $\mathfrak{h}$ in $\mathcal{A}_{\leq d}$, a Lie subalgebra that fits into the exact sequence

$$0\to\mathcal{A}_{\leq d-1}\to\tilde{\mathfrak{h}}\to\mathfrak{h}\to 0.$$

The ideal $\mathcal{A}_{\leq d-1}$ is nilpotent b/c $\deg[\cdot,\cdot]\leq -d$. So the SES splits. This gives a locally finite representation of $\mathfrak{h}$ in $\mathcal{A}$ by derivations. It lifts to a rational representation of a simply connected cover $\tilde{H}$ of H on $\mathcal{A}$ by automorphisms. Note that, by the construction, $\operatorname{gr}\mathcal{A}\simeq A$ is $\tilde{H}$-equivariant. Since $\tilde{H}\curvearrowright A$ factors through H, the same is true for $\tilde{H}\curvearrowright\mathcal{A}$. $\square$