Lecture 21.

1) Filtered quantizations of Y

Refs: [BPW], Sec 3; Sec 17 in [E]

1.0) Reminder: We take a s/simple group G & pick a parabolic subgroup $P=L\ltimes U$. Pick an L-equivariant cover $\tilde{\mathbb{O}}_L$ of a nilpotent orbit in $\mathfrak{l}^*$. Assume $\tilde{\mathbb{O}}_L$ is birationally rigid $\Leftrightarrow X_L$ is $\mathbb{Q}$-factorial & terminal. In Sec 2 of Lec 20 we've stated that $\mathbb{C}[X_L]$ admits a unique filtered quantization $\mathcal{A}_L$, that the action of L on X_L lifts to $\mathcal{A}_L$ and the lift has a quantum comoment map $\Phi: \mathfrak{l}\to \mathcal{A}_{L,\leqslant d}$ s.t. $\operatorname{gr}\Phi: \mathfrak{l}\to \mathbb{C}[X_L]_d$ coincides w. μ^*: $\mathfrak{l}\to\mathbb{C}[X_L]_d$. We normalize Φ by requiring $\Phi|_{\mathfrak{z}(\mathfrak{l})}=0$.

Consider the diagonal action $L\curvearrowright T^*(G/U)\times X_L$ w. comoment map $\mu_L^*: \xi\mapsto \xi_{G/U}\otimes 1+1\otimes\mu^*$. Then $\Phi_L: \xi\mapsto \xi_{G/U}\otimes 1+1\otimes \Phi$ is a quantum comoment map for $L\curvearrowright \mathcal{D}_{G/U}\otimes\mathcal{A}_L$ (a sheaf on G/U). Note that Φ_L lifts μ_L^*. We will take a quantum Hamiltonian reduction for shifts of Φ_L by characters of $\mathfrak{l}$.

1.1) Construction: This requires a normalization. Let $\rho_{G/P}$ be one half of the character of L in $\Lambda^{top}(\mathfrak{g}/\mathfrak{p})^*$. E.g. for $P=B$, $\rho_{G/B}$ is $\frac{1}{2}\sum_\alpha \alpha$, where α runs over the positive roots. In Lie theory this element is commonly denoted by ρ. Later on, we will comment on the importance of the shift.

Now, for $\lambda \in \mathfrak{z} := (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$, we define the sheaf of filtered algebras $\mathcal{D}_\lambda$ on G/P as the quantum Hamiltonian reduction

$$(\mathcal{D}_{G/U} \otimes \mathcal{A}_L^\hbar) /\!/\!/_{\lambda - \rho_{G/P}} L$$

The conditions of Lemma from Sec 2.3 of Lec 19 are satisfied, so $\mathcal{D}_\lambda$ is a filtered quantization of $\pi_* \mathcal{O}_Y$ (where π is a projection $Y = \mu_L^{-1}(0)/L$ & $\pi: Y \twoheadrightarrow G/P$ is a projection)

Example: If $X_L = \{0\}$, then $\mathcal{D}_\lambda = \mathcal{D}_{G/P}^{\lambda - \rho_{G/P}}$ by Theorem in Sec 1.0 of Lec 20. In particular, note that $-\rho_{G/P} = \frac{1}{2} c_1(K_{G/P})$, so, for $\lambda = 0$, we get differential operators in "one half of the canonical bundle."

Rem: Recall that we have also considered the universal classical Hamiltonian reduction $Y_\mathfrak{z} := \mu_L^{-1}(\mathfrak{z})/\!/L$, a flat scheme/ $\mathfrak{z}$. Similarly, we can consider its quantum counterpart $\mathcal{D}_\mathfrak{z}$ defined by $\mathcal{D}_{\mathfrak{z}} := (\mathcal{D}_{G/U}\otimes \mathcal{A}_L / [\mathcal{D}_{G/U}\otimes \mathcal{A}_L]\,\varphi([\mathfrak{l},\mathfrak{l}]))^L$. We remark that $\mathfrak{l}/[\mathfrak{l},\mathfrak{l}]$ maps to $\mathcal{D}_{\mathfrak{z}}$ via φ_L. The image is central: $[\varphi_L(\xi), a] = 0$ if a is L-invariant. So $\mathcal{D}_{\mathfrak{z}}$ is a sheaf (on G/P) of filtered $\mathbb{C}[\mathfrak{z}]$-algebras. Since the functor of taking L-invariants is exact, $\mathcal{D}_\lambda = \mathbb{C}_{\lambda-\rho_{G/P}} \otimes_{\mathbb{C}[\mathfrak{z}]} \mathcal{D}_{\mathfrak{z}}$. And arguing as in Sec 2.3 of Lec 19, we see that $\mathcal{D}_{\mathfrak{z}}$ is a filtered quantization of $Y_{\mathfrak{z}}$.

1.2) (Derived) global sections (addi'l reading: Sec 3 in [BPW])

Let $\tilde{\mathbb{O}}$ be the open G-orbit in Y, a G-equivariant cover of a nilpotent orbit. Set $X = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$. By 3) of Theorem in Sec 2 of Lec 14, we have that the partial resolution morphism (now denoted by ϖ) $\varpi: Y \to X$ gives $\varpi^*: \mathbb{C}[X] \xrightarrow{\sim} \mathbb{C}[Y]$. And by Sec 2 of Lec 12, we have $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$.

Theorem: $\Gamma(\mathcal{D}_\lambda)$ is a filtered quantization of $\mathbb{C}[X] = \mathbb{C}[Y]$. Moreover, $H^i(G/P, \mathcal{D}_\lambda) = 0$ for $i > 0$.

Proof: Since $\pi: Y \to G/P$ is affine, we have $H^i(G/P, \pi_* \mathcal{O}_Y) = 0$

$\forall\, i>0$. Let $(\pi_* \mathcal{O}_Y)_i$ denote the ith graded component of $(\pi_*\mathcal{O}_Y)_i$, so that we have a SES

$$0 \to \mathcal{D}_{\lambda, \leq i-1} \to \mathcal{D}_{\lambda,\leq i} \to (\pi_*\mathcal{O}_Y)_i \to 0, \ \forall i \quad (1)$$

This yields a long exact sequence in cohomology:

$$\to H^j(\mathcal{D}_{\lambda,\leq i-1}) \to H^j(\mathcal{D}_{\lambda,\leq i}) \to H^j((\pi_*\mathcal{O}_Y)_i) = 0$$

So we use the induction on i (w. the base of $i=-1$, where $\mathcal{D}_{\lambda,\leq -1}=0$) to show that $H^j(\mathcal{D}_{\lambda,\leq i})=0\ \forall i$, while

$$0 \to H^0(\mathcal{D}_{\lambda,\leq i-1}) \to H^0(\mathcal{D}_{\lambda,\leq i}) \to H^0((\pi_*\mathcal{O}_Y)_i) \to 0 \quad (2)$$

is exact. The cohomology commutes w. direct limits (Prop 2.9 in Ch. 3 of Hartschorne's book) so $H^j(\mathcal{D}_\lambda)=0$ & (2) shows $\operatorname{gr}\Gamma(\mathcal{D}_\lambda) \xrightarrow{\sim} \mathbb{C}[Y]$. Since the isomorphism $\operatorname{gr}\mathcal{D}_\lambda \xrightarrow{\sim} \pi_*\mathcal{O}_Y$ is Poisson, so is $\operatorname{gr}\Gamma(\mathcal{D}_\lambda) \xrightarrow{\sim} \mathbb{C}[Y]$. This shows that $\Gamma(\mathcal{D}_\lambda)$ is indeed a filtered quantization of $\mathbb{C}[Y]$. $\square$

One can deduce that $H^j(Y_{\mathfrak{z}},\mathcal{O})=0\ \forall j>0$ from $H^j(Y,\mathcal{O})=0$ - left as an exercise, compare to Prop. in Sec 2 of Lec 15 (and use its argument to prove that $H^j(Y_{\mathfrak{z}'},\mathcal{O})=0$ for all subspaces $\mathfrak{z}'\subset\mathfrak{z}$ using the induction on $\dim \mathfrak{z}'=0$). From here one repeats the argument of Thm to deduce that $\Gamma(\mathcal{D}_{\mathfrak{z}})$

is a filtered quantization of $\mathbb{C}[Y_{\mathfrak{z}}]$. And then the same argument as in Theorem shows that $H^j(G/P, \mathcal{D}_{\mathfrak{z}})=0 \ \forall j>0$.

Lemma: $\Gamma(\mathcal{D}_\lambda) = \mathbb{C}_{\lambda-\rho_{G/P}} \otimes_{\mathbb{C}[\mathfrak{z}]} \Gamma(\mathcal{D}_{\mathfrak{z}})$.

Proof: The scheme $Y_{\mathfrak{z}}$ is flat over $\mathfrak{z}$, Rem in Sec 1.1 of Lec 14. So is the sheaf $\pi_* \mathcal{O}_{Y_{\mathfrak{z}}}$ and hence its graded components. Using exact sequences similar to (1), we see that $\mathcal{D}_{\mathfrak{z}}$ is flat$/\mathfrak{z}$.

Let $f_1,\dots f_k$ be a basis in the space of affine functions on $\mathfrak{z}$ that vanish on $\lambda-\rho_{G/P}$. We can form the Koszul complex ([E], Ch. 17) for these functions viewed as sections of $\mathcal{D}_{\mathfrak{z}}$. The terms of this complex are $\mathcal{D}_{\mathfrak{z}}^{\oplus ?}$ and, thx to the flatness the homology is $\mathcal{D}_\lambda$. Note that $R\Gamma(\mathcal{D}_{\mathfrak{z}}), R\Gamma(\mathcal{D}_\lambda)$ are in cohomological deg 0. We conclude that the homology of the Koszul complex for $f_1,\dots f_k \in \Gamma(\mathcal{D}_{\mathfrak{z}})$ is $\Gamma(\mathcal{D}_\lambda)$ in deg 0 & vanish in deg >0. In particular, $\mathbb{C}_{\lambda-\rho} \otimes_{\mathbb{C}[\mathfrak{z}]} \Gamma(\mathcal{D}_{\mathfrak{z}}) = \Gamma(\mathcal{D}_\lambda)$ $\square$

Rem: It turns out that $\Gamma(\mathcal{D}_{\mathfrak{z}})$ is a free $\mathbb{C}[\mathfrak{z}]$-module. This will appear in the next (& last!) homework.

1.3) Quantum Hamiltonian action.

We claim that there is a Hamiltonian G-action on $\mathcal{D}_\lambda$ & $\Gamma(\mathcal{D}_\lambda)$ (and also on $\mathcal{D}_\mathfrak{z}$ & $\Gamma(\mathcal{D}_\mathfrak{z})$). We start by observing that G acts on $D_{G/U}\otimes\mathcal{A}_L$ (on the 1st factor) w. quantum comoment map $\Phi_G(\eta):=\eta_{G/U}\otimes 1$. This action commutes w. L & $\Phi_L(\xi)$ is G-invariant. It follows that the G-action descends to $\mathcal{D}_\lambda$. Similarly, $\Phi_G(\eta)$ is L-invariant and so gives an element $\underline{\Phi}_G(\eta)\in\Gamma(\mathcal{D}_\lambda)$. From $[\Phi_G(\eta), a]=\eta.a$ for any local section a of $D_{G/N}\otimes\mathcal{A}_L$ we deduce that $[\underline{\Phi}_G(\eta),\underline{a}]=\eta.\underline{a}$ for any local section $\underline{a}$ of $\mathcal{D}_\lambda$. So $G\curvearrowright\mathcal{D}_\lambda$ is Hamiltonian w. quantum comoment map $\underline{\Phi}_G$. Also note that passing to the associated graded sheaves we recover the Hamiltonian action on $\pi_*\mathcal{O}_Y$ that comes from the Hamiltonian G-action on Y.

Then we can pass to the global sections and get a Hamiltonian G-action on $\Gamma(\mathcal{D}_\lambda)$ which lifts that on $\mathbb{C}[X]$. In particular, we get an algebra homomorphism

$$\Phi_G:\mathcal{U}(\mathfrak{g})\longrightarrow\Gamma(\mathcal{D}_\lambda)$$

(formerly denoted by $\underline{\Phi}_G$ - but we simplify the notation).

1.4) Example: $Y = T^*(G/B)$.

Recall, Lec 11, that we identify the center of $\mathcal{U}(\mathfrak{g})$ w. $\mathbb{C}[\mathfrak{h}^*]^W$ by means of the Harish-Chandra isomorphism. So for $\lambda \in \mathfrak{h}^*$ we can view the maximal ideal $\mathfrak{m}_\lambda \subset \mathbb{C}[\mathfrak{h}^*]^W$ as sitting inside the center and form the quotient $\mathcal{U}_\lambda = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{m}_\lambda$. We have seen in Lec 11 that it comes w. a natural filtration turning it into a filtered quantization of $\mathbb{C}[\mathcal{N}]$.

Thm: $\varphi_G : \mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(\mathcal{D}_\lambda)$ factors through $\mathcal{U}_\lambda \xrightarrow{\sim} \Gamma(\mathcal{D}_\lambda)$.

Proof: Step 1: We claim that φ_G is surjective. This is b/c $\operatorname{gr}\varphi_G : \operatorname{gr}\mathcal{U}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \longrightarrow \operatorname{gr}\Gamma(\mathcal{D}_\lambda) = \mathbb{C}[\mathcal{N}]$ is the classical comoment map. It's just the restriction map, hence surjective $\Rightarrow \varphi_G$ is surjective.

Step 2: We claim that $\varphi_G(Z) = \mathbb{C}\,(\subset \Gamma(\mathcal{D}_\lambda))$. Indeed, $Z = \mathcal{U}(\mathfrak{g})^G$. The map φ_G is G-equivariant so $\varphi_G(Z) \subset \Gamma(\mathcal{D}_\lambda)^G$. But $\operatorname{gr}\Gamma(\mathcal{D}_\lambda)^G = \mathbb{C}[\mathcal{N}]^G = [\mathcal{N}$ contains a dense G-orbit$] = \mathbb{C}$. So $\Gamma(\mathcal{D}_\lambda)^G = \mathbb{C}$. We conclude that φ_G factors through $\mathcal{U}(\mathfrak{g}) \twoheadrightarrow \mathcal{U}_{\lambda'} \twoheadrightarrow \Gamma(\mathcal{D}_\lambda)$ for some λ'. On the level of associated

graded, both $\mathcal{U}(\mathfrak{g}) \twoheadrightarrow \mathcal{U}_{\lambda'}$ & $\mathcal{U}(\mathfrak{g}) \twoheadrightarrow \Gamma(\mathcal{D}_\lambda)$ give $\mathbb{C}[\mathfrak{g}^*] \twoheadrightarrow \mathbb{C}[\mathcal{N}]$.
So $\operatorname{gr} \mathcal{U}_{\lambda'} \xrightarrow{\sim} \operatorname{gr} \Gamma(\mathcal{D}_\lambda)$ & hence $\mathcal{U}_{\lambda'} \xrightarrow{\sim} \Gamma(\mathcal{D}_\lambda)$.
We need to show

$$\lambda' \in W\lambda \qquad (*)$$

Step 3: We start by checking $(*)$ when $\lambda-\rho$ is integral dominant. For this we will show that $\Gamma(\mathcal{D}_\lambda)$ acts on the irreducible $\mathfrak{g}$-module $V_{\lambda-\rho}$ w. highest wt. $\lambda-\rho$ so that the induced action of $\mathcal{U}(\mathfrak{g})$ is the usual one. By the construction of the HC isomorphism the center acts on $V_{\lambda-\rho}$ by the evaluation at λ (to be denoted ev_λ), and so $(*)$ will follow.

The sheaf $\pi_* \mathcal{D}_{G/U}$ acts on $\pi_* \mathcal{O}_{G/U}$. The elements of the form $\xi_{G/U} - \langle \chi, \xi \rangle$ for $\xi \in \mathfrak{h}$, $\chi \in \mathfrak{X}(T)$ act by 0 on

$$(\pi_* \mathcal{O}_{G/U})^{T,\chi} = \{ \sigma \mid t.\sigma = \chi(t)\sigma,\ \forall\, t \in T \} \qquad (1)$$

This is left as an exercise. So $\mathcal{D}_{\chi+\rho}$ acts on this sheaf (on G/B). The action of $\varphi_G(\xi) \in \Gamma(\mathcal{D}_{\chi+\rho})$ comes from the G-equivariant structure on (1). Now we note that (1) is just $\mathcal{O}(\chi)$, which implies our claim: $\Gamma(\mathcal{O}(\lambda-\rho)) \xrightarrow[G]{\sim} V_{\lambda-\rho}$ by the Borel-Weil Thm.

Step 4: Now we check (*) for arbitrary λ. The action of G on $\mathcal{D}_{\mathfrak{z}}$ and hence on $\Gamma(\mathcal{D}_{\mathfrak{z}})$ is Hamiltonian as well, moreover the composition $\mathcal{U}(\mathfrak{g}) \xrightarrow{\varphi_G} \Gamma(\mathcal{D}_{\mathfrak{z}}) \twoheadrightarrow \Gamma(\mathcal{D}_\lambda)$ is a quantum comoment map. Restricting φ_G to the G-invariants we get $\varphi_G: \mathcal{U}(\mathfrak{g})^G \longrightarrow \Gamma(\mathcal{D}_{\mathfrak{z}})^G$. The image $\mathbb{C}[\mathfrak{z}] \hookrightarrow \Gamma(\mathcal{D}_{\mathfrak{z}})$ consists of G-invariants. Similarly to Step 2, we have

Exercise: Show that $\Gamma(\mathcal{D}_{\mathfrak{z}})^G = \mathbb{C}[\mathfrak{z}]$. Hint: every fiber of $Y_{\mathfrak{z}} \to \mathfrak{z}$ contains a dense G-orbit. Deduce that $\mathbb{C}[\mathfrak{z}] \xrightarrow{\sim} \mathbb{C}[Y_{\mathfrak{z}}]^G$ and use $\operatorname{gr} \Gamma(\mathcal{D}_{\mathfrak{z}}) \xrightarrow{\sim} \mathbb{C}[Y_{\mathfrak{z}}]$ to complete the proof.

Now note that the map $\mathbb{C}[\mathfrak{h}^*]^W = \mathcal{U}(\mathfrak{g})^G \longrightarrow Z(\Gamma(\mathcal{D}_\lambda)) = \mathbb{C}$ factors through

$$\mathbb{C}[\mathfrak{h}^*]^W \xrightarrow[\text{alg. homom.}]{} \mathbb{C}[\mathfrak{h}^*] \xrightarrow{ev_\lambda} \mathbb{C} \qquad (2)$$

(2) coincides w ev_λ for all λ s.t. $\lambda-\rho$ is dominant & integral by Step 3. Since the set of such weights is Zariski dense in $\mathfrak{h}^*$, it follows that (1) coincides w. ev_λ for all $\lambda \in \mathfrak{h}^*$ (exercise). □