Lecture 22

1) Classification of quantizations.

1.0) Introduction: Let $G\supset P=L\ltimes U$, $\widetilde{\mathbb{O}}_L$, X_L, $Y=\operatorname{Ind}_P^G(X_L)$, $\mathfrak{z}:=(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ be as in the previous lecture. We assume that X_L is $\mathbb{Q}$-factorial & terminal (hence so is Y, Sec 1.2 in Lec 16).

In Lec 21, we have constructed a family of filtered quantizations of $\pi_*\mathcal{O}_Y$ ($\pi: Y\twoheadrightarrow G/P$) parameterized by points of $\mathfrak{z}$: $\lambda\mapsto\mathcal{D}_\lambda$. It turns out that this gives a complete classification.

Thm: Every filtered quantization of $\pi_*\mathcal{O}_Y$ is isomorphic to $\mathcal{D}_\lambda$ for exactly one $\lambda\in\mathfrak{z}$.

We already know this theorem in a special case: $X_L=\{0\}$ so $Y=T^*(G/P)$. Here it follows from the classification of sheaves of TDO on a smooth variety Y_0: by $H^2(\Omega_{Y_0}^{\geq 1})$.

Below we'll state a classification result for "formal quantizations" of smooth symplectic varieties Y° w. $H^i(Y^\circ, \mathcal{O})=0$ for $i=1,2$. We'll deduce Thm from there.

1.1) Formal quantizations.

For now, let Y be a Poisson scheme.

Definition: A **formal quantization** of Y is a sheaf of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{D}_\hbar$ on Y (in Zariski topology) together w. an isomorphism $\iota: \mathcal{D}_\hbar/(\hbar) \xrightarrow{\sim} \mathcal{O}_Y$ of sheaves of algebras s.t.

(1) $\hbar$ is not a zero divisor in $\mathcal{D}_\hbar$. In particular, $\mathcal{D}_\hbar/(\hbar)$ comes w. a Poisson bracket: thx to ι, we have $[a,b]\in(\hbar)$ for any local sections a,b of $\mathcal{D}_\hbar$, and we set

$$\{a+(\hbar), b+(\hbar)\} = \frac{1}{\hbar}[a,b]+(\hbar).$$

(2) $\mathcal{D}_\hbar$ is complete & separated in the $\hbar$-adic topology: $\mathcal{D}_\hbar \xrightarrow{\sim} \varprojlim \mathcal{D}_\hbar/(\hbar^n)$.

(3) And ι is a Poisson isomorphism.

Rem: For $k\geqslant 2$, it makes sense to talk about **kth truncated quantizations** of Y. These are sheaves of $\mathbb{C}[\hbar]/(\hbar^k)$-algebras

$\mathcal{D}_{\hbar,k}$ on Y that are flat over $\mathbb{C}[\hbar]/(\hbar^k) \rightsquigarrow \{\cdot,\cdot\}$ on $\mathcal{D}_{\hbar,k}/(\hbar)$ w. ι satisfying (3). If $\mathcal{D}_\hbar$ is a formal quantization of Y, then $\mathcal{D}_\hbar/(\hbar^k)$ is a kth truncated quantization. Conversely, if $\mathcal{D}_{\hbar,k}$ is a family of kth truncated quantizations w. $\mathcal{D}_{\hbar,k+1}/(\hbar^k) \xrightarrow{\sim} \mathcal{D}_{\hbar,k}$, then $\varprojlim \mathcal{D}_{\hbar,k}$ is a formal quantization.

1.2) The case of affine Y.

Suppose Y is affine. Formal quantizations of $\mathbb{C}[Y]$ were introduced in Lec 3. A connection between the formal quantizations of Y and of $\mathbb{C}[Y]$ is parallel to that between affine schemes & algebras of regular functions.

Below is a somewhat informal discussion of the correspondence between (truncated) quantizations of algebras & schemes. More details will be in a complement note.

- Every truncated quantization $\mathcal{A}_{\hbar,k}$ of $A=\mathbb{C}[Y]$ can be localized to a sheaf of $\mathbb{C}[\hbar]/(\hbar^k)$-algebras, $Loc(\mathcal{A}_{\hbar,k})$ on Y. Then $Loc(\mathcal{A}_{\hbar,k})$ is a kth truncated quantization of Y.
- If $\mathcal{D}_{\hbar,k}$ is a kth truncated quantization of Y, then $\Gamma(\mathcal{D}_{\hbar,k})$ is a kth truncated quantization of $\mathbb{C}[Y]$.

Now we send a formal quantization $\mathcal{A}_\hbar$ of $\mathbb{C}[Y]$ to $Loc(\mathcal{A}_\hbar) := \varprojlim Loc(\mathcal{A}_{\hbar,k})$ & a formal quantization $\mathcal{D}_\hbar$ of Y to $\Gamma(\mathcal{D}_\hbar)$. These procedures are mutually inverse to each other.

1.4) Graded formal quantizations.

Recall that in the setting of algebras, one can talk about graded formal quantizations (by HW1, for $\mathbb{Z}_{\geqslant 0}$-graded algebras A those are in bijection w. filtered quantizations of A). This extends to quantizations of schemes.

Assume that $\mathbb{C}^\times \curvearrowright Y$ rescaling $\{\cdot,\cdot\}$ by $t \mapsto t^{-d}$ ($d \in \mathbb{Z}_{>0}$). Assume also that the following condition on Y holds:

(*) $\forall y \in Y$ $\exists$ $\mathbb{C}^\times$-stable open affine nghd of y.

This is not very restrictive: this holds if Y is normal, a theorem of Sumihiro.

Definition: A grading on a formal quantization $(\mathcal{D}_\hbar, \iota)$ is an action of $\mathbb{C}^\times$ on $\mathcal{D}_\hbar$ by $\mathbb{C}$-algebra automorphism w. $t.\hbar = t^d \hbar$ s.t $\iota: \mathcal{D}_\hbar/(\hbar) \longrightarrow \mathcal{O}_Y$ is equivariant & $\mathbb{C}^\times \curvearrowright \mathcal{D}_\hbar$ is rational in the following sense:

$\forall$ open $\mathbb{C}^\times$-stable affine U & $\forall$ $k>0 \Rightarrow \mathbb{C}^\times \curvearrowright \Gamma(U,\mathcal{D}_\hbar)/(\hbar^k)$ is rational.

Remark: Let's get back to our $Y = \mathrm{Ind}_P^G(X_L)$. Let $\mathcal{D}$ be a filtered quantization of $\pi_* \mathcal{O}_Y$ (a sheaf on G/P). We can consider the completed Rees sheaf $\hat{R}_h(\mathcal{D})$ (still on G/P). We get a formal quantization of Y doing the procedure from the previous section on π^{-1}(open affines) & gluing) modulo a caveat: in the Rees sheaf $\deg h = 1$ and in our case $\deg \hbar = 2$. This can be fixed in a number of ways: there's a distinguished subsheaf of $\mathbb{C}[h^2]$-subalgebras $\hat{R}_{h^2}(\mathcal{D}) \subset \hat{R}_h(\mathcal{D})$ w. $\mathbb{C}[[h]] \otimes_{\mathbb{C}[[h^2]]} \hat{R}_{h^2}(\mathcal{D}) \xrightarrow{\sim} \hat{R}_h(\mathcal{D})$ & so $\hat{R}_{h^2}(\mathcal{D})$ is a graded formal quantization (w. $\hbar := h^2$). Or we can modify the definition of a formal quantization to allow $[\mathcal{D}_\hbar, \mathcal{D}_\hbar] \subset \hbar^d \mathcal{D}_\hbar$. We are going to ignore this caveat. Our conclusion is that filtered quantizations of $\pi_* \mathcal{O}_Y$ are in bijection w. graded formal quantizations of Y.

1.5) Classification.

Suppose Y is symplectic. Bezrukavnikov & Kaledin (Section 4 in [BK]) have defined the noncommutative period map

$$\mathrm{Per}: \{\text{formal quantizations of } Y\}/\text{iso} \xrightarrow{\sim} H^2_{DR}(X)[[\hbar]]$$

w. $\mathrm{Per}|_{\hbar=0} = [\omega]$, where ω is the symplectic form on Y & $[\omega]$ is its cohomology class. Note that if $\mathbb{C}^\times \curvearrowright Y$ rescaling ω w. nonzero character, we have $[\omega]=0$. The following theorem gives a classification of all formal quantizations.

Thm 1 (special case of Thm 1.8 in [BK]): Suppose $H^i(Y, \mathcal{O}_Y)=0$ for $i=1,2$. Then Per is an embedding w. image $[\omega]+\hbar H^2_{DR}(Y)$.

For graded formal quantizations, we have the following

Thm 2 ([L4], Section 2.3) Under the same assumption, Per gives a bijection between $\{$graded formal quant'ns of $Y\}/\text{iso}$ & $\hbar H^2_{DR}(Y) \subset (\hbar H^2_{DR}(Y)[[\hbar]])$.

Let's give an example of computation of period. Let $Y = \mathrm{Ind}_P^G(X_L)$. It's not smooth but Y^{reg} is symplectic. Let $\mathcal{D}_{\lambda,\hbar}$ be the graded formal quantization of Y corresponding to $\mathcal{D}_\lambda$ (see

Sec 1.4). Consider it's restriction $\mathcal{D}^{reg}_{\lambda,\hbar}$ to Y^{reg}. We want to compute its period. Recall that $\operatorname{Pic}(Y)$ is identified w. $\mathfrak{X}(L)$ (Sec 1.2 in Lec 16). For $\lambda\in\mathfrak{X}(L)$, we write $c_1(\lambda)\in H^2_{DR}(Y^{reg})$ for $c_1(\mathcal{O}_Y(\lambda)|_{Y^{reg}})$. Extend $\lambda\mapsto c_1(\lambda)$ to $\mathfrak{z}=\mathfrak{X}(L)\otimes_{\mathbb{Z}}\mathbb{C}\longrightarrow H^2_{DR}(Y^{reg})$. It's an isomorphism (Sec. 1.1 of Lec 17).

Thm 3 ([L4], Sec 5.4; [BPW], Sec 3.4) $\operatorname{Per}(\mathcal{D}^{reg}_{\lambda,\hbar})=c_1(\lambda)$.

Recall that $H^i(Y^{reg},\mathcal{O})=0$ for $i=1,2$, Sec 2 of Lec 12. So we get the following corollary of Thms 2 & 3.

Corollary: $\mathfrak{z}\xrightarrow{\sim}\{$graded formal quantizations of $Y^{reg}\}$ via $\lambda\mapsto\mathcal{D}^{reg}_{\lambda,\hbar}$

1.6) Quantizations of Y.

Now $Y=\operatorname{Ind}_P^G(X_L)$ as in Sec 1.0.

It turns out that (graded) formal quantizations of Y & of Y^{reg} are in bijection. We'll state a result in the ungraded setting. Let η denote the inclusion $Y^{reg}\hookrightarrow Y$.

Theorem: The push-forward ι_* and the pullback ι^* (of sheaves of $\mathbb{C}[[\hbar]]$-modules) define mutually inverse bijections between quantizations of Y^{reg} & Y.

This combined w. Corollary from Sec. 1.5 imply the theorem from Sec 1.0.

Proof: Everything but the claim that for a quantization $\mathcal{D}_\hbar^{reg}$ of Y^{reg}, $\eta_*\mathcal{D}_\hbar$ satisfies $(\eta_*\mathcal{D}_\hbar^{reg})/(\hbar) \xrightarrow{\sim} \mathcal{O}_Y$ is a routine check (left as an exercise). Set $\mathcal{D}_{\hbar,k}^{reg} := \mathcal{D}_\hbar^{reg}/(\hbar^k)$. $\forall k$ we have a SES $0 \to \mathcal{D}_{\hbar,k-1}^{reg} \xrightarrow{\hbar} \mathcal{D}_{\hbar,k}^{reg} \to \mathcal{O}_{Y^{reg}} \to 0$. Apply $R\eta_*$ getting a long exact sequence:

$$0 \to \eta_*\mathcal{D}_{\hbar,k-1}^{reg} \xrightarrow{\hbar} \eta_*\mathcal{D}_{\hbar,k}^{reg} \to \underset{\substack{\| \\ \mathcal{O}_Y}}{\eta_*\mathcal{O}_{Y^{reg}}} \to R^1\eta_*\mathcal{D}_{\hbar,k-1}^{reg} \xrightarrow{\hbar} R^1\eta_*\mathcal{D}_{\hbar,k}^{reg} \to \underset{\substack{\| \\ 0}}{R^1\eta_*\mathcal{O}_{Y^{reg}}}$$

$\longleftarrow$ by Sec 2 of Lec 12 $\longrightarrow$

Using induction on k (compare to Sec 1.2 of Lec 21), we get $R^1\eta_*\mathcal{D}_{\hbar,k}=0$, hence

$$0 \to \eta_*\mathcal{D}_{\hbar,k-1}^{reg} \xrightarrow{\hbar} \eta_*\mathcal{D}_{\hbar,k}^{reg} \to \mathcal{O}_Y \to 0 \qquad (1)$$

is exact. Note that ι_* & $\varprojlim$ commute so

$$\eta_*\mathcal{D}_\hbar^{reg} = \eta_*(\varprojlim \mathcal{D}_{\hbar,k}^{reg}) \xrightarrow{\sim} \varprojlim \eta_*\mathcal{D}_{\hbar,k}^{reg} \qquad (2)$$

(1) shows that $\eta_* \mathcal{D}^{reg}_{\hbar,k}$ is a kth truncated quantization, while the kernel of $\eta_* \mathcal{D}^{reg}_{\hbar,k} \xrightarrow{\hbar^{k-1}} \eta_* \mathcal{D}^{reg}_{\hbar,k}$ is $\hbar^{k-1}\mathcal{O}_Y$. It follows that the r.h.s. of (2) is a formal quantization of Y $\square$