Lecture 23

1) Recap & goals
2) Automorphisms & isomorphisms.

Ref: [L1], Secs 2.5, 3.6, 3.7.

1) Recap & goals.

What we want: an isomorphism of three sets:

(i) G-equivariant covers of (co)adjoint G-orbits.

(ii) Filtered Poisson deformations of graded Poisson algebras of the form $\mathbb{C}[\tilde{\mathbb{O}}]$, where $\tilde{\mathbb{O}}$ is a G-equivariant cover of a nilpotent orbit. They are viewed up to a filtered Poisson algebra isomorphism (less restrictive that an isomorphism of filtered Poisson deformations).

(iii) Similar to (ii) but for filtered quantizations.

A bijection between (i) & (iii) is our algebraic Orbit method. And we'll discuss (i) $\xleftrightarrow{\sim}$ (ii) & (ii) $\xleftrightarrow{\sim}$ (iii).

Here's a bunch of things we have covered.

(I) We have stated there's a universal graded Poisson deformation $X_{\mathfrak{h}_X/W_X}$ of a conical symplectic singularity X. Here $\mathfrak{h}_X := H^2(Y^{reg}, \mathbb{C})$, where Y is a $\mathbb{Q}$-factorial terminalization of X, and W_X is a reflection group in $GL(\mathfrak{h}_X)$ - we haven't explained how it is constructed. In the case when $X = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$, we have $Y = \operatorname{Ind}_P^G(X_L)$ w. $X_L = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}_L]$ w. minimal L. Note that Y depends on the choice of P. We have $\mathfrak{h}_X = \mathfrak{z}\ (= (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*)$. We can consider the universal deformation $Y_{\mathfrak{z}}$ of Y over $\mathfrak{z}$. Then $X_{\mathfrak{z}} = \operatorname{Spec} \mathbb{C}[Y_{\mathfrak{z}}]$ is the base change $\mathfrak{z} \times_{\mathfrak{h}_X/W_X} X_{\mathfrak{h}_X/W_X}$ for the quotient morphism $\mathfrak{z} = \mathfrak{h}_X \longrightarrow \mathfrak{h}_X/W_X$. See Lec 18, Sec 1.1.

(II) We note that $G \curvearrowright Y, Y_{\mathfrak{z}}, X_{\mathfrak{z}}$ - Hamiltonian actions. We also note that we have a Hamiltonian action on $X' :=$ $\operatorname{Spec} \mathcal{A}^0$, where $\mathcal{A}^0$ is <u>any</u> filtered Poisson deformation of $A := \mathbb{C}[\tilde{\mathbb{O}}]$, Sec 1.1 of Lec 17. The action is unique up to an automorphism from $\exp\{\mathcal{A}^0_{\leq 1}, \cdot\}$: if $\tilde{\varphi}$ is the preimage of $\mathfrak{g}$ in $\mathcal{A}^\circ_{\leq 2}$ under $\mathcal{A}^\circ_{\leq 2} \twoheadrightarrow A_2 (\hookleftarrow \mathfrak{g})$, then the SES $0 \to \mathcal{A}^\circ_{\leq 1} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ splits (Levi's thm) & different splittings are conjugate by an element of $\exp\{\mathcal{A}^\circ_{\leq 1}, \cdot\}$ (Mal'cev's thm). Note that $\exp\{\mathcal{A}^\circ_{\leq 1}, \cdot\}$ acts on $\mathcal{A}^\circ$ by automorphisms of a filtered Poisson deformations as this action is the identity on $\operatorname{gr}\mathcal{A}^\circ$).

The action of G on $\operatorname{Spec}(\mathcal{A}^\circ)$ has an open orbit that is a cover of a coadjoint orbit, this has been established in Sec 1.2 of Lec 17. This gives a map (ii) $\to$ (i) above.

(III) Conversely, for every cover $\tilde{\mathbb{O}}'$ of an adjoint orbit the algebra $\mathbb{C}[\tilde{\mathbb{O}}']$ carries a filtration making it a filtered Poisson deformation of a suitable $\mathbb{C}[\tilde{\mathbb{O}}]$. If $\tilde{\mathbb{O}}'$ covers a nilpotent orbit, then we take $\tilde{\mathbb{O}} := \tilde{\mathbb{O}}'$. In general, $\tilde{\mathbb{O}}'$ is induced, $\tilde{\mathbb{O}}' = \operatorname{Ind}_L^G(\tilde{\mathbb{O}}_L, \chi)$ and we can assume L is minimal - by transitivity of induction (so $\tilde{\mathbb{O}}_L$ is birationally rigid). Then we take $\tilde{\mathbb{O}} = \operatorname{Ind}_L^G(\tilde{\mathbb{O}}_L)$. The filtration on $\mathbb{C}[\tilde{\mathbb{O}}']$ comes from $\mathbb{C}[\tilde{\mathbb{O}}'] = \mathbb{C}[Y_\chi] = \mathbb{C}[Y_{\mathbb{C}\chi}]/(z-1)\mathbb{C}[Y_{\mathbb{C}\chi}]$ (see Sec 2 of Lec 15). This gives a map (i) $\to$ (ii).

Rem: The claim that (i) $\to$ (ii) $\to$ (i) is the identity follows b/c $\widetilde{\mathbb{O}}'$ is the open orbit in Spec $\mathbb{C}[\widetilde{\mathbb{O}}']$. Now, for a filtered Poisson deformation $\mathcal{A}^\circ$ of A we need to establish a filtered Poisson isomorphism $\mathcal{A}^\circ \xrightarrow{\sim} \mathbb{C}[\widetilde{\mathbb{O}}']$ w. filtration on the target as in (III). A G-equivariant isomorphism is easy: it's the pullback under the inclusion $\widetilde{\mathbb{O}}' \hookrightarrow$ Spec $\mathcal{A}^\circ$. Our later analysis will show that it respects the filtrations, establishing (i) $\xleftrightarrow{\sim}$ (ii)

(IV): We have also constructed a family of quantizations parameterized by λ: $\Gamma(\mathcal{D}_\mathfrak{z})$, a filtered quantization of $\mathbb{C}[X_\mathfrak{z}]$ and its specialization $\Gamma(\mathcal{D}_\lambda) = \mathbb{C}_\lambda \otimes_{\mathbb{C}[\mathfrak{z}]} \Gamma(\mathcal{D}_\mathfrak{z})$. It turns out that these exhaust all quantizations.

We will be interested in a number of related questions:

(a) How to construct the Weyl group W_X?

(b) For which $\lambda, \lambda' \in \mathfrak{z}$ filtered Poisson deformations $\mathbb{C}[X_\lambda]$, $\mathbb{C}[X_{\lambda'}]$ are isomorphic as filtered Poisson algebras (thx to (II) one can choose this isomorphism to be also G-equivariant).

(c) Why $\Gamma(\mathcal{D}_{\mathfrak{z}})$ is independent of the choice of P (w. fixed L) and why W_X acts on $\Gamma(\mathcal{D}_{\mathfrak{z}})$.

(d) For which λ, λ': $\Gamma(\mathcal{D}_\lambda), \Gamma(\mathcal{D}_{\lambda'})$ are isomorphic as filtered algebras. Similarly to (b), this isomorphism can be chosen to be G-equivariant.

We will see that there's a subgroup $\tilde{W}_X \subset GL(\mathfrak{z})$ containing W_X s.t. the answers to <u>both</u> (b) & (d) is: iff $\lambda' \in \tilde{W}_X \lambda$ ($\lambda' \in W_X \lambda$ is equivalent to $\Gamma(\mathcal{D}_\lambda), \Gamma(\mathcal{D}_{\lambda'})$ being isomorphic as filtered quantizations - and similarly for filtered Poisson deformations). It turns out that $\Gamma(\mathcal{D}_\lambda)$'s exhaust all quantizations. So our characterization of isomorphisms of filtered quantizations/filtered Poisson deformations will give a bijection (ii) $\xleftrightarrow{\sim}$ (iii) thereby establishing the algebraic Orbit method.

2) Automorphisms & isomorphisms.

2.1) Graded Poisson automorphisms.

Let X be a conical symplectic singularity. By $\mathrm{Aut}(X)$ we denote the group of graded Poisson automorphisms of $\mathbb{C}[X]$. It's algebraic, it embeds as a closed subgroup into $\prod_{i=1}^{\ell} GL(\mathbb{C}[X]_i)$, where ℓ is chosen in such a way that $\bigoplus_{i=1}^{\ell}\mathbb{C}[X]_i$ generates $\mathbb{C}[X]$. If G is an algebraic group with a fixed homomorphism to $\mathrm{Aut}(X)$ we consider the group $\mathrm{Aut}_G(X)\subset \mathrm{Aut}(X)$ of G-equivariant elements in $\mathrm{Aut}(X)=$ (the centralizer of the image of G).

Examples: 1) Let $\Gamma\subset Sp(V)$ be a finite group. The group $N_{Sp(V)}(\Gamma)/\Gamma$ naturally acts on V/Γ, faithfully & by graded Poisson automorphisms & so embeds into $\mathrm{Aut}(V/\Gamma)$. Conversely, using some kind of Galois theory, one can show that any element of $\mathrm{Aut}(V/\Gamma)$ lifts to an element of $N_{Sp(V)}(\Gamma)$. Hence $N_{Sp(V)}(\Gamma)/\Gamma \xrightarrow{\sim} \mathrm{Aut}(V/\Gamma)$.

2) Let $X=\mathrm{Spec}\,\mathbb{C}[\widetilde{\mathbb{O}}]$ for a G-equivariant cover $\widetilde{\mathbb{O}}$ of

a nilpotent orbit. We want to compute $\mathrm{Aut}_G(X)$. Note that $\mathrm{Aut}_G(X)$ preserves $\widetilde{\mathbb{O}} \subset X$. Pick a point $x \in \widetilde{\mathbb{O}}$ so that $\widetilde{\mathbb{O}} = G/G_x$. The group of G-equivariant automorphisms is $N_G(G_x)/G_x$ acting by $n.(gG_x) = gn^{-1}G_x$. If such an automorphism is symplectic, it must preserve the moment map $\mu: \widetilde{\mathbb{O}} \to \mathfrak{g}^*$ (G is s/simple so μ is unique) meaning $\mu(gG_x) = \mu(gn^{-1}G_x)$ $\forall g$. So $nG_x \in \mathrm{Aut}_G(X) \Rightarrow n \in Z \cap N_G(G_x) = N_Z(G_x)$, $e := \mu(x)$ & $Z := Z_G(e)$ ($\supset G_x$ as finite index subgroup). Indeed, $\mu(x) = \mu(n^{-1}x) = n^{-1}\mu(x)$ $\Rightarrow n^{-1} \in Z$.

Conversely, we claim that for $n \in N_Z(G_x)$, the map $gG_x \mapsto gn^{-1}G_x$ is in $\mathrm{Aut}_G(X)$. This map preserves hence the symplectic form, $\mu^*\omega_{KK}$, on $\widetilde{\mathbb{O}}$. Also recall that the $\mathbb{C}^\times$-action on $\widetilde{\mathbb{O}}$ is by $t.gG_x = g\gamma(t)^{-1}G_x$, where $\gamma: \mathbb{C}^\times \to G$ is the 1-parameter subgroup w. $d_1\gamma = h$ (from $\mathfrak{sl}_2$-triple (e,h,f)) Since $G_x \supset Z_G(e)^\circ$ we can choose a representative for any element of $Z_G(e)/G_x$ in $Q := Z_G(e,h,f)$. And for all $n \in Q$ the map $gG_x \mapsto gn^{-1}G_x : \widetilde{\mathbb{O}} \to \widetilde{\mathbb{O}}$ is $\mathbb{C}^\times$-equivariant. Any automorphism of $\widetilde{\mathbb{O}}$ lifts to X thx to $\mathbb{C}[\widetilde{\mathbb{O}}] = \mathbb{C}[X]$. So, $\mathrm{Aut}_G(X) \xleftarrow{\sim} N_Z(G_x)/G_x$, a finite group.

2.2) $\mathrm{Aut}_G(X)$ & filtered automorphisms.

Here we explain why we should consider $\mathrm{Aut}_G(X)$ in relation w. the question about isomorphisms of filtered deformations.

The first observation is very basic:

Lemma: Let $(\mathcal{A},\iota)$, $(\mathcal{A}',\iota')$ be filtered quantizations of $\mathbb{C}[X] =: A$ (where, recall $\iota: \mathrm{gr}\mathcal{A} \xrightarrow{\sim} A$ is a graded Poisson isomorphism). Let $\varphi: \mathcal{A} \xrightarrow{\sim} \mathcal{A}'$ be a filtered algebra isomorphism. Then $\underline{\varphi} := \iota' \circ \mathrm{gr}\varphi \circ \iota^{-1}: A \to A$ is in $\mathrm{Aut}(X)$. The same conclusion holds for filtered Poisson deformations & filtered Poisson isomorphisms.

Proof is an exercise. Note also that $\underline{\varphi} = id \Leftrightarrow \varphi$ is an isomorphism of filtered quantizations.

On the other hand, note that $\mathrm{Aut}_G(X)$ acts on the set $\{$filt. quant'ns of $\mathbb{C}[X]\}/\mathrm{iso}$: $\gamma.(\mathcal{A},\iota) := (\mathcal{A}, \gamma\circ\iota)$. And $\mathcal{A}, \mathcal{A}'$ are isomorphic as filtered algebras $\Leftrightarrow (\mathcal{A},\iota)$ & $(\mathcal{A}',\iota')$ lie in the same $\mathrm{Aut}(X)$-orbit. The similar claim holds for filtered Poisson deformations.

Rem: The action of $\mathrm{Aut}(X)^\circ$ on the set of isomorphism classes is actually trivial. Here's a sketch of the proof. We have $H^1_{DR}(X^{reg})=0$ so every graded Poisson derivation of $\mathbb{C}[X]$ is inner ([L1], Sec. 2.5). So it lifts to a filtration preserving derivation of every filtered quantization (and to a filtered Poisson derivation of every filtered Poisson deformation). From here one deduces the claim (exercise).