Lecture 24.

1) Automorphisms & isomorphisms, cont'd.

1.0) Intro.

Let $\widetilde{\mathbb{O}}$ be a G-equivariant cover of a nilpotent G-orbit $\mathbb{O}$. Let $L, \widetilde{\mathbb{O}}_L$ be s.t. $\widetilde{\mathbb{O}} = \operatorname{Ind}_L^G(\widetilde{\mathbb{O}}_L)$ & L is minimal. As usual, set $X = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}]$, $X_L = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}_L]$

We are going to answer the following questions:

- How to describe the Namikawa-Weyl group W_X in Lie-theoretic terms?

- How to compute the action of $\operatorname{Aut}_G(X) (= N_{Z_G(e)}(G_x)/G_x$, $x \in \widetilde{\mathbb{O}}$, $e = \mu(x)$ – see Sec 2.1 of Lec 23) on $\mathfrak{h}_X/W_X$. This will tell us when two filtered deformations are isomorphic as filtered Poisson algebras), see Sec. 2.2 of Lec 23.

We start now & finish in the next lecture.

Consider the group $N_G(L) \subset G$. It acts on L, so on $\mathfrak{l}$ & $\mathfrak{l}^*$. Let $\mu: X_L \to \mathfrak{l}^*$ be the moment map. Consider the group

$$N_G(L,\widetilde{\mathbb{O}}_L) := \{(n,\zeta)\in N_G(L)\times \mathrm{Aut}(X_L) \mid \mathrm{Ad}(n)\circ\mu = \mu\circ\zeta\}.$$

Exercise: • $L \lhd N_G(L,\widetilde{\mathbb{O}}_L)$ via $\ell\mapsto(\ell,\ell)$.

• If $\widetilde{\mathbb{O}}_L = \mathbb{O}_L$, then $N_G(L,\widetilde{\mathbb{O}}_L) \xrightarrow{\sim} \{n\in N_G(L) \mid \mathrm{Ad}(n)\mathbb{O}_L = \mathbb{O}_L\}$.

• $\ker[N_G(L,\widetilde{\mathbb{O}}_L)\to N_G(L)] \xleftarrow{\sim} \mathrm{Aut}_L(X_L)$ via $\zeta\mapsto(1,\zeta)$.

Since $\mathrm{Aut}_L(X_L)$ & $N_G(L)/L$ are finite, so is $N_G(L,\widetilde{\mathbb{O}}_L)/L =: \widetilde{W}_X$.

Here's the main result for this & next lecture.

Thm: We have a SES of groups

$$1\to W_X\to \widetilde{W}_X\to \mathrm{Aut}_G(X)\to 1$$

1.1) $\widetilde{W}_X \curvearrowright X_{\mathfrak{z}}$.

Note that $\widetilde{W}_X$ acts on $\mathfrak{z}$ via the projection $\widetilde{W}_X\to N_G(L)/L$. We are going to produce an action of $\widetilde{W}_X$ on $X_{\mathfrak{z}}$ by $G\times\mathbb{C}^\times$-equivariant Poisson automorphisms preserving the moment map and lifting the action on $\mathfrak{z}$. We will also see that the variety $X_{\mathfrak{z}} = \mathrm{Spec}\,\mathbb{C}[Y_{\mathfrak{z}}]$ is independent of the choice of P.

Pick $(n,\zeta)\in N_G(L,\tilde{\mathbb{O}}_L)$. Pick a parabolic subgroup $P = L\ltimes U$. Set ${}^nU := nUn^{-1}$; ${}^nP := L\ltimes {}^nU$ is also a parabolic. We write ${}^nY = \mathrm{Ind}_{{}^nP}^G(X_L)$ and ${}^nY_{\mathfrak{z}}$ for its deformed version.

Step 1: we are going to produce a $G\times\mathbb{C}^\times$-equivariant Poisson isomorphism $Y_{\mathfrak{z}} \longrightarrow {}^nY_{\mathfrak{z}}$ making the following diagram commutative:

$$\begin{tikzcd} Y_{\mathfrak{z}} \arrow[r] \arrow[d] & {}^nY_{\mathfrak{z}} \arrow[d] \\ \mathfrak{z} \arrow[r, "n"] & \mathfrak{z} \end{tikzcd}$$

Note that $g\mapsto gn^{-1}$ gives rise to $G/U \xrightarrow{\sim} G/{}^nU$, hence to $T^*(G/U)\xrightarrow{\sim} T^*(G/{}^nU)$, to be denoted by n. Then we get an isomorphism $(n,\zeta): T^*(G/U)\times X_L \longrightarrow T^*(G/{}^nU)\times X_L$.

Exercise: It descends to a $G\times\mathbb{C}^\times$-equiv't iso $Y_{\mathfrak{z}} \xrightarrow{\sim} {}^nY_{\mathfrak{z}}$, $[g,\alpha,x]\mapsto [gn^{-1}, n.\alpha, \zeta.x]$, intertwining moment maps to $\mathfrak{g}^*$.

Step 2: this is the main part: let $P' = L\ltimes U'$, and $Y', Y'_{\mathfrak{z}}$ be the corresponding varieties. Let $X_{\mathfrak{z}} := \mathrm{Spec}\,\mathbb{C}[Y_{\mathfrak{z}}]$, $X'_{\mathfrak{z}} := \mathrm{Spec}\,\mathbb{C}[Y'_{\mathfrak{z}}]$. We claim that there's a $G\times\mathbb{C}^\times$-equivariant Poisson isomorphism

$X_\mathfrak{z} \xrightarrow{\sim} X'_\mathfrak{z}$ intertwining the maps to $\mathfrak{g}^* \times \mathfrak{z}$.

Recall that under the isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ coming from the Killing form $\mathfrak{z}$ gets identified w. $\mathfrak{z}(\mathfrak{l})$. Define

$$\mathfrak{z}^\circ := \{X \in \mathfrak{z} \mid \mathfrak{z}_\mathfrak{g}(X) = \mathfrak{l} \ (\Leftrightarrow G_X = L)\}$$

This is a Zariski open subset, in fact, it's the complement to finitely many hyperplanes: pick a Cartan $\mathfrak{h} \subset \mathfrak{l}$, then the hyperplanes are $\ker \alpha|_{\mathfrak{z}(\mathfrak{l})}$ for roots α w. $\mathfrak{g}_\alpha \not\subset \mathfrak{l}$.

Set $Y^\circ_\mathfrak{z} = \mathfrak{z}^\circ \times_\mathfrak{z} Y_\mathfrak{z}$. We start w. identifying $Y^\circ_\mathfrak{z} \xrightarrow{\sim} Y'^\circ_\mathfrak{z}$.

Exercise 1 (compare Sec 2.2 of Lec 14, Sec. 1.3 of Lec 15)

Construct an isomorphism $G \times^L (\mathfrak{z}^\circ \times X_L) \xrightarrow{\sim} Y^\circ_\mathfrak{z}$ (a key step:

$$U \times (\mathfrak{z}^\circ \times X_L) \longrightarrow \{(\alpha, x) \in (\mathfrak{g}/\mathfrak{h})^* \times X_L \mid \alpha|_\mathfrak{l} - \mu(x) \in \mathfrak{z}^\circ\}$$

$(u, \beta, x) \mapsto (u(\beta + \mu(x)), x)$ is an iso). Furthermore, show that it is $G \times \mathbb{C}^\times$-equivariant and intertwines natural maps to $\mathfrak{g}^* \times \mathfrak{z}^\circ$.

Applying the same construction to $Y'_\mathfrak{z}$ we get

$$Y^\circ_\mathfrak{z} \xrightarrow{\sim} G \times^L (\mathfrak{z}^\circ \times X_L) \xrightarrow{\sim} Y'^\circ_\mathfrak{z}$$

The composition is $G \times \mathbb{C}^\times$-equivariant, intertwines the maps to $\mathfrak{g}^* \times \mathfrak{z}^\circ$ and hence is Poisson (exercise).

Observe that $Y_\mathfrak{z} \to \mathfrak{g}^* \times \mathfrak{z}$ is projective: it's the composition of the finite morphism

$$Y_\mathfrak{z} \to (G \times^P \mathfrak{g}^*) \times \mathfrak{z},\ [g,(\alpha,x)] \mapsto ([g,\alpha], \alpha|_\mathfrak{l} - \mu(x))\ \&$$

the projection $(G\times^P \mathfrak{g}^*)\times \mathfrak{z} \to \mathfrak{g}^* \times \mathfrak{z}$, $([g,\alpha], x) \mapsto (g\alpha, x)$.

So we can uniquely extend the open embedding $Y^0_\mathfrak{z} \xrightarrow{\sim} Y'^0_\mathfrak{z} \hookrightarrow Y'_\mathfrak{z}$ to a morphism from an open subset of $Y_\mathfrak{z}$ w. complement of codim $\geqslant 2$ (compare to Sec. 1.3 of Lec 17). We can do the same in the other direction. Since we are extending mutually inverse isomorphisms $Y^0_\mathfrak{z} \leftrightarrows Y'^0_\mathfrak{z}$ we get mutually inverse isomorphisms between open subsets of $Y_\mathfrak{z}, Y'_\mathfrak{z}$ w. complements of codim $\geqslant 2$. Then we apply the Hartogs theorem to get $X_\mathfrak{z} \xrightarrow{\sim} X'_\mathfrak{z}$. This isomorphism is $G\times\mathbb{C}^\times$-equivariant and intertwines the morphisms to $\mathfrak{g}^*\times\mathfrak{z}$ (exercise).

Step 3: We show that the isomorphisms $X_\mathfrak{z} \xrightarrow{(n,\zeta)} X_\mathfrak{z}$ induced by $Y_\mathfrak{z} \xrightarrow[\sim]{(n,\zeta)} {}^nY_\mathfrak{z}$, $[g,\alpha,x] \mapsto [gn^{-1}, n.\alpha, \zeta x]$, give an action of $\widetilde{W}_X$ on $X_\mathfrak{z}$. For this notice that the isomorphism above lifts $n: \mathfrak{z}\to\mathfrak{z}$, hence restricts to $Y^0_\mathfrak{z} \xrightarrow{\sim} {}^nY^0_\mathfrak{z}$.

Exercise: Under the identification $\dot{Y}^{?\circ}_{\mathfrak{z}} \xrightarrow{\sim} G\times^L(\mathfrak{z}^\circ\times X_L)$ (Exer 1 in Step 2), the isomorphism (n,ζ) becomes

$$[(g,\beta,x)] \longmapsto [gn^{-1}, n.\beta, \zeta x].$$

Deduce that these isomorphisms indeed give an action of $\widetilde{W}_X = N_G(L,\widetilde{\mathbb{O}}_L)$ on $G\times^L(\mathfrak{z}^\circ\times X_L)$.

Now to deduce that the isomorphisms (n,ζ) constitute an action of $\widetilde{W}_X$ on $X_{\mathfrak{z}}$ we observe that $\dot{Y}^{?}_{\mathfrak{z}}\to X_{\mathfrak{z}}$ restricts to $\dot{Y}^{?\circ}_{\mathfrak{z}} \xrightarrow{\sim} X^\circ_{\mathfrak{z}}$. Indeed, we have the Stein decomposition $\dot{Y}^{?}_{\mathfrak{z}} \longrightarrow X_{\mathfrak{z}} \longrightarrow \mathfrak{g}^*\times\mathfrak{z}$ & $\dot{Y}^{?\circ}_{\mathfrak{z}} \longrightarrow \mathfrak{g}^*\times\mathfrak{z}^\circ$ is finite. The claim that the isomorphisms (n,ζ) give an action of $\widetilde{W}_X$ amounts to checking that they agree w. compositions and the elements (ℓ,ℓ) act trivially. It's enough to check both claims on Zariski dense subsets & $X^\circ_{\mathfrak{z}}\subset X_{\mathfrak{z}}$ is Zariski dense. Now use Exercise.

1.2) Hamiltonian isomorphisms.

In fact the isomorphisms $X_\chi \xrightarrow{(n,\zeta)} X_{n\chi}$ w. $\chi\in\mathfrak{z}^\circ$ & $(n,\zeta)\in N_G(L,\widetilde{\mathbb{O}}_L)$ can be characterized conceptually and this will play an important role in proving Thm from Sec 1.0.

Prop: Let $\varphi: X_\chi \to X_{\chi'}$, $\chi, \chi' \in \mathfrak{z}_1^\circ$ be a Hamiltonian isomorphism, i.e. a G-equivariant map intertwining the moment maps to $\mathfrak{g}^*$ (and hence a Poisson isomorphism b/c $X_\chi, X_{\chi'}$ contain open G-orbit - but we are not going to use this). Then φ is given by a unique element $(n, \zeta) \in N_G(L, \tilde{\mathbb{O}}_L)/L$.

Proof: Note that $X_\chi = G \times^L X_L$ w. moment map $[g, x] \mapsto g(\chi + \mu(x))$. Pick $x \in X_L$. Let $\varphi([1,x]) = [n^{-1}, x']$, $n \in G$, $x' \in X_L$. Since φ intertwines the moment maps, $\chi + \mu(x) = n^{-1}(\chi' + \mu(x'))$. Take s/simple parts $\chi = n^{-1}\chi' \Rightarrow L = Z_G(\chi) = n^{-1} Z_G(\chi') n = n^{-1} L n \Rightarrow n \in N_G(L)$.

Next, φ restricts to $X_L \simeq \mu^{-1}(\chi + \tilde{\mathbb{O}}_L) \to \mu^{-1}(\chi' + \tilde{\mathbb{O}}_L) \simeq X_L$. Let ζ be the restriction. We claim that $(n, \zeta) \in N_G(L, \tilde{\mathbb{O}}_L)$. We have

$$\zeta(\ell x) = (n \ell n^{-1}) \zeta(x) \ \& \ \mu(\zeta(x)) = Ad(n)\mu(x), \forall \ell \in L, x \in \tilde{\mathbb{O}}_L \quad (*)$$

(exercise). The conjugation w. n is a $\mathbb{C}^\times$-equivariant symplectomorphism of $\mathbb{O}_L$. Thx to $(*)$, ζ preserves $\tilde{\mathbb{O}}_L \subset X_L$ and we claim that $(*)$ also implies that ζ is a $\mathbb{C}^\times$-equivariant symplectom'm of $\tilde{\mathbb{O}}_L$, and so $\zeta \in Aut(X_L) \Rightarrow (n, \zeta) \in N_G(L, \tilde{\mathbb{O}}_L)$.

The form on $\tilde{\mathbb{O}}_L$ is $\mu^*(\omega_{KK})$, we deduce that $\zeta: \tilde{\mathbb{O}}_L \to \tilde{\mathbb{O}}_L$ is a symplectomorphism from (*) & the claim that $Ad(n): \mathbb{O}_L \to \mathbb{O}_L$ is a symplectomorphism. We write eu for the vector field coming from the $\mathbb{C}^\times$-action. We have that ζ is $\mathbb{C}^\times$-equiv't $\Leftrightarrow \zeta^* eu_{\tilde{\mathbb{O}}_L} = eu_{\tilde{\mathbb{O}}_L}$ (pullback under an etale morphism). But $eu_{\tilde{\mathbb{O}}} = \mu^* eu_{\mathbb{O}}$, and $\mu \circ \zeta = Ad(n) \circ \mu$ & $Ad(n)^* eu_{\mathbb{O}_L} = eu_{\mathbb{O}_L}$, which implies $\zeta^* eu_{\tilde{\mathbb{O}}_L} = eu_{\tilde{\mathbb{O}}_L}$ thx to (*).

Under the identification $X_x \simeq G \times^L X_L$, (n,ζ) acts (last Exer. in Sec 1.1) by $(n,\zeta).[g,x] = [gn^{-1}, \zeta x]$. On the other hand, $\varphi([g,x]) = g\varphi[1,x] = g[n^{-1}, \zeta x] = [gn^{-1}, \zeta x]$. So φ coincides w. (n,ζ).

Now we need to show the uniqueness: if

$$[gn_1^{-1}, n_1 X, \zeta_1 x] = [gn_2^{-1}, n_2 X, \zeta_2 x] \text{ for } \forall g \in G, x \in X_L,$$

then $(n_1,\zeta_1)L = (n_2,\zeta_2)L$. This is left as an exercise. $\square$