Lecture 25

1) Automorphisms & isomorphisms, finished.

2) Action on quantization.

1.0) Recap.

$\tilde{\mathbb{O}} = \mathrm{Ind}_L^G(\tilde{\mathbb{O}}_L)$, $X := \mathrm{Spec}\ \mathbb{C}[\tilde{\mathbb{O}}]$, $X_L = \mathrm{Spec}\ \mathbb{C}[\tilde{\mathbb{O}}_L]$.

$N_G(L, \tilde{\mathbb{O}}_L) = \{(n,\zeta) \in N_G(L) \times \mathrm{Aut}(X_L) \mid n \circ \mu = \mu \circ \zeta\} \rightsquigarrow \tilde{W}_X = N_G(L, \tilde{\mathbb{O}}_L)/L$

$Y_\mathfrak{z} = G \times^P \{(\alpha, x) \in (\mathfrak{g}/\mathfrak{n})^* \times X_L \mid \alpha|_\mathfrak{l} - \mu(x) \in \mathfrak{z}\} \rightsquigarrow X_\mathfrak{z} := \mathrm{Spec}\ \mathbb{C}[Y_\mathfrak{z}]$.

Last time we've seen that:

(A) $\tilde{W}_X \curvearrowright X_\mathfrak{z}$ by $G \times \mathbb{C}^\times$-equivariant automorphisms intertwining moment maps & making $X_\mathfrak{z} \to \mathfrak{z}$ equivariant.

(B) for $\chi \in \mathfrak{z}^\circ = \{\chi \in \mathfrak{z} \mid G_\chi = L\}$ any G-equivariant isomorphism $X_\chi \to X_{\chi'}$ intertwining the moment maps is the action by an element of $\tilde{W}_X$.

Our goal is to prove the following theorem.

Theorem: Have SES: $1 \to W_X \to \tilde{W}_X \to \mathrm{Aut}_{\mathfrak{C}}(X) \to 1$.

Note that we indeed have a homomorphism $\tilde{W}_X \longrightarrow \mathrm{Aut}_{\mathfrak{C}}(X)$: by restricting the $\tilde{W}_X \curvearrowright X_{\mathfrak{z}}$ from (A) to $X = \{0\} \times_{\mathfrak{z}} X_{\mathfrak{z}}$.

So what we need to do:

- Produce an embedding $W_X \hookrightarrow \tilde{W}_X$ whose image is the kernel of $\tilde{W}_X \twoheadrightarrow \mathrm{Aut}_{\mathfrak{C}}(X)$.
- Show that the homomorphism $\tilde{W}_X \to \mathrm{Aut}_{\mathfrak{C}}(X)$ is surjective.

Both use (B) above.

1.1) Embedding $W_X \hookrightarrow \tilde{W}_X$.

Recall the universal deformation $X_{\mathfrak{h}_X/W_X}$. We have $\mathfrak{z} \xrightarrow{\sim} \mathfrak{h}_X$ &

$$X_{\mathfrak{z}} \xrightarrow{\sim} \mathfrak{z} \times_{\mathfrak{h}_X/W_X} X_{\mathfrak{h}_X/W_X} \qquad (1)$$

$\mathbb{C}^\times$-equiv. Poisson iso, Sec 1.1 of Lec 18. This gives $W_X \curvearrowright X_{\mathfrak{z}}$ (by $\mathbb{C}^\times$-equiv. Poisson isomorphisms).

Hamiltonian actions extend to deformations (Sec 1.1 of Lec 17)

In particular, we have one of $X_{\mathfrak{h}_X/W_X}$, then it lifts to $\mathfrak{z} \times_{\mathfrak{h}_X/W_X} X_{\mathfrak{h}_X/W_X}$

Lemma: The Hamiltonian action on $X_{\mathfrak{z}}$ extending that on X commuting w. $\mathbb{C}^\times$ & making $X_{\mathfrak{z}} \to \mathfrak{z}$ invariant is unique.

Proof: Similarly to what was explained in (II) of Sec 1 of Lec 23, two such actions are conjugate by an automorphism of the form $\exp(\{f,\cdot\})$ w. $\deg f = 2$ & $f|_X = 0$ ($\Leftrightarrow f \in \mathfrak{z}^* \mathbb{C}[X_{\mathfrak{z}}]$). Since the degree of $\mathfrak{z}^* \subset \mathbb{C}[X_{\mathfrak{z}}]$ is 2 & $\mathbb{C}[X_{\mathfrak{z}}]$ is positively graded, we deduce that $f \in \mathfrak{z}^* \Rightarrow \exp(\{f,\cdot\}) = id$ $\square$

So (1) is G-equivariant & intertwines the moment maps.

Now we are going to produce an embedding $W_X \hookrightarrow \tilde{W}_X$.

Note that the actions of W_X & $\tilde{W}_X$ on $X_{\mathfrak{z}}$ are faithful: for W_X this follows from the construction. An element of $\tilde{W}_X$ is uniquely determined by its restriction to X_x, $x \in \mathfrak{z}^\circ$ - the uniqueness part of the proposition in Sec 1.2 of Lec 24.

So the action of $\tilde{W}_X$ is faithful as well. To get an embedding $W_X \hookrightarrow \tilde{W}_X$ it remains to show that W_X acts on $X_{\mathfrak{z}}$ by transformations from $\tilde{W}_X$. Thx to Prop'n in Sec 1.2 of Lec 24, this will follow if we check that for Zariski generic $\chi \in \mathfrak{z}$, $w: X_\chi \longrightarrow X_{w\chi}$ is a G-equivariant isomorphism intertwining the moment maps (details of the reduction are left as an exercise). But this follows b/c the G-action & the moment map are lifted from $X_{\mathfrak{z}_X/W_X}$. So we get an embedding $W_X \hookrightarrow \tilde{W}_X$.

We now claim that $W_X \subset \tilde{W}_X$ coincides w. the kernel of $\tilde{W}_X \longrightarrow Aut_G(X)$. The inclusion $W_X \subset \ker$ comes from the construction of $W_X \curvearrowright X_{\mathfrak{z}} = \mathfrak{z} \times_{\mathfrak{z}_X/W_X} X_{\mathfrak{z}_X/W_X}$ w. $W_X \curvearrowright X_{\mathfrak{z}_X/W_X}$ trivial.

Now let $u \in \ker[\tilde{W}_X \longrightarrow Aut_G(X)]$. Note that $\tilde{W}_X$ acts on $\mathbb{C}[X_{\mathfrak{z}}]$ by graded Poisson algebra automorphisms. For $\chi \in \mathfrak{z}$, $u: \mathbb{C}[X_\chi] \longrightarrow \mathbb{C}[X_{u\chi}]$ is an iso of filtered Poisson algebras and, thx to $u \in \ker$, of filtered Poisson deformations. By the classification of those $\exists\ w_\chi \in W_X$ w. $u\chi = w_\chi \chi$. Since this holds for all χ, $\exists\ w \in W_X$ w. $u\chi = w\chi$. Replacing u w. $w^{-1}u$ we can assume that u acts trivially on both X & $\mathfrak{z}$.

Exercise: Show that u acts as a unipotent operator on each graded component of $\mathbb{C}[X_{\mathfrak{z}}]$. Then use that $\tilde{W}_X$ is finite, to conclude $u = id$.

1.2) Epimorphism $\tilde{W}_X \to \mathrm{Aut}_G(X)$.

Let $\varphi \in \mathrm{Aut}_G(X)$. We claim that φ lifts to a $G\times\mathbb{C}^\times$-equivariant automorphism of $X_{\mathfrak{z}_X/W_X}$ intertwining the moment maps to $\mathfrak{g}^*$. Namely, let $X'_{\mathfrak{z}_X/W_X}$ be another graded Poisson deformation that coincides w. $X_{\mathfrak{z}_X/W_X}$ as a scheme but the identification $\{0\}\times_{\mathfrak{z}_X/W_X} X'_{\mathfrak{z}_X/W_X} \xrightarrow{\sim} X$ is twisted by φ. By the universal property of $X_{\mathfrak{z}_X/W_X}$ applied to $X'_{\mathfrak{z}_X/W_X}$ we get an automorphism $\tilde{\varphi}$ of $X_{\mathfrak{z}_X/W_X}$ of graded Poisson variety lifting φ.

Exercise: Similarly to Lemma in Sec 1.2, show that $\tilde{\varphi}$ is G-equivariant & intertwines the moment maps to $\mathfrak{g}^*$.

Since $W_X \triangleleft \tilde{W}_X$ by Sec 1.1, $\tilde{W}_X/W_X$ acts on $X_{\mathfrak{z}}/W_X = X_{\mathfrak{z}_X/W_X}$. We need to show $\tilde{\varphi}$ lies in the image of this action. It's sufficient to show that the restriction of $\tilde{\varphi}$ to a Zariski generic fiber of $X_{\mathfrak{z}_X/W_X} \to \mathfrak{z}_X/W_X$ coincides w. that of an element of $\tilde{W}_X$. This is done as in the previous section and is left as an exercise.

Remark: Recall (Corollary in Sec. 1.6 of Lec 16) that the filtered Poisson deformations of $\mathbb{C}[X]$ are parameterized by pts in $\mathfrak{h}_X/W_X$. The action of $\mathrm{Aut}_G(X)$ on $\mathfrak{h}_X/W_X$ coming from $\mathrm{Aut}_G(X) \simeq \widetilde{W}_X/W_X$ coincides w. the action on isomorphism classes of Poisson deformations, Sec. 2.2. of Lec 23, by the construction.

1.3) Filtered Poisson deformations vs orbit covers.

In Lecture 23 we had sets (i) - equivariant covers of coadjoint G-orbits - and (ii) - filtered Poisson deformations of $\mathbb{C}[\widetilde{\mathbb{O}}]$. We had maps (ii) $\to$ (i), taking the open orbit in $\mathrm{Spec}\,\mathcal{A}^\circ$, and (i) $\to$ (ii): sending $\widetilde{\mathbb{O}}' = \mathrm{Ind}_L^G(\widetilde{\mathbb{O}}_L, \chi)$ to $\mathbb{C}[X_\chi]$.

Lemma: The composition (ii) $\to$ (i) $\to$ (ii) is the identity.

Sketch of proof:

Every $\mathcal{A}^\circ$ is $\mathbb{C}[X_{\chi'}]$ for some $\chi' \in \mathfrak{z}$ (as a filtered algebra). As argued in Remark in Sec 1 of Lec 23, we have a G-equivariant isomorphism $\mathbb{C}[X_{\chi'}] \to \mathbb{C}[X_\chi]$ intertwining the moment maps. If both $\chi, \chi' \in \mathfrak{z}^\circ$, then the isomorphism is given by an action of an element of $\widetilde{W}_X$, such isomorphisms are filtration preserving. On the other hand, if $X=X'=0$, then the argument of Example in Sec 2.1, shows that a Hamiltonian automorphism of $\mathbb{C}[X]$ preserves the grading. The general case interpolates between the two: we can assume $Z_G(X)=Z_G(X')$, denote this Levi by M. We have a cover $\widetilde{\mathbb{O}}_M$ of a nilpotent orbit in $\mathfrak{m}^*$ s.t. $X_X=\mathbb{C}[\operatorname{Ind}_Q^G(\widetilde{X}_M,X)]$, $X_{X'}=\mathbb{C}[\operatorname{Ind}_Q^G(\widetilde{X}_M,X')]$ (here Q is a parabolic w. Levi M, compare to Sec 1.3 of Lec 15). One then shows that the isomorphism comes from the action of the group $N_G(M,\widetilde{\mathbb{O}}_M)$ defined similarly to $N_G(L,\widetilde{\mathbb{O}}_L)$. Details are left as a hard exercise. $\square$

2) Action on quantization.

Recall the quantization $\mathcal{D}_z$ of Y_z from Lec 21. Here's our main result about $\Gamma(\mathcal{D}_z)$.

Theorem: (1) $\Gamma(\mathcal{D}_z)$ is independent of P.

(2) $\widetilde{W}_X$ acts on $\Gamma(\mathcal{D}_z)$ by filtered algebra auto-

morphisms. The action has the following properties:

(a) On $\mathrm{gr}\,\Gamma(\mathcal{D}_{\mathfrak{z}}) = \mathbb{C}[X_{\mathfrak{z}}]$ it coincides w. the action constructed in Lec 24.

(b) $\mathbb{C}[\mathfrak{z}] \longrightarrow \Gamma(\mathcal{D}_{\mathfrak{z}})$ is $\widetilde{W}_X$-equivariant.

(c) The action of $\widetilde{W}_X$ is by G-equivariant automorphisms intertwining the quantum comoment maps.

We'll discuss the proof in the next (and final) lecture. For now we remark that this theorem completes the proof of the bijection (ii) $\longleftrightarrow$ (iii) mentioned in Lec 23 modulo the claim that filtered quantizations are parameterized by points of $\mathfrak{h}_X/W_X$ (and are the specializations of $\Gamma(\mathcal{D}_{\mathfrak{z}})$). Indeed the action of $\mathrm{Aut}_G(X)$ on the set of isomorphism classes of filtered Poisson deformations is the action of $\mathrm{Aut}_G(X)$ on $\mathfrak{h}_X/W_X$ coming from the SES in Thm in Sec 1.0. Thx to Theorem above (& property (b), in particular), the same is true for quantizations.