Lecture 26 (a.k.a. the last).

1) Automorphisms & isomorphisms of quantizations.

2) What would be covered if this were a 2 semester class?

1.0) Recap.

Our goal here is to explain what goes into the proof of the theorem stated last time.

Theorem: (1) $\Gamma(\mathcal{D}_\mathfrak{z})$ is independent of P.

(2) $\widetilde{W}_X$ acts on $\Gamma(\mathcal{D}_\mathfrak{z})$ by filtered algebra automorphisms. The action has the following properties:

(a) On $\operatorname{gr}\Gamma(\mathcal{D}_\mathfrak{z}) = \mathbb{C}[X_\mathfrak{z}]$ it coincides w. the action constructed in Lec 24.

(b) $\mathbb{C}[\mathfrak{z}] \longrightarrow \Gamma(\mathcal{D}_\mathfrak{z})$ is $\widetilde{W}_X$-equivariant.

(c) The action of $\widetilde{W}_X$ is by G-equivariant automorphisms intertwining the quantum comoment maps.

We'll see that 1) is the crucial part.

1.1) Independence of P.

The proof is based on two results. The first one is geometric. Let $Y'_\mathfrak{z}$ denote the deformed induced variety for a parabolic P'. Recall projective morphisms $Y_\mathfrak{z} \to X_\mathfrak{z} \leftarrow Y'_\mathfrak{z}$.

Proposition 1: There are open subvarieties $Y^2_\mathfrak{z} \subset Y$, $Y'^2_\mathfrak{z} \subset Y'_\mathfrak{z}$ s.t.

- $\operatorname{codim}_{Y_\mathfrak{z}} Y_\mathfrak{z} \setminus Y^2_\mathfrak{z}$, $\operatorname{codim}_{Y'_\mathfrak{z}} Y'_\mathfrak{z} \setminus Y'^2_\mathfrak{z} \geqslant 3$.
- there's an isomorphism $Y^2_\mathfrak{z} \to Y'^2_\mathfrak{z}$ intertwining morphisms to $X_\mathfrak{z}$.

Proof: Recall the open subsets $Y^\circ_\mathfrak{z} = \mathfrak{z}^\circ \times_\mathfrak{z} Y_\mathfrak{z} \subset Y_\mathfrak{z}$, $Y'^\circ_\mathfrak{z} \subset Y_\mathfrak{z}$, $X^\circ_\mathfrak{z} \subset X_\mathfrak{z}$. We have $Y^\circ_\mathfrak{z} \xrightarrow{\sim} X^\circ_\mathfrak{z} \xleftarrow{\sim} Y'^\circ_\mathfrak{z}$ (Sec 1.1 of Lec 24). Also, $\forall x \in \mathfrak{z}$, the morphism $Y_x \to X_x$ is an isomorphism over the open orbit in X_x (which is $\tilde{\mathbb{O}}_x$, see Sec 1.2 of Lec 17). Set

$$X^1_\mathfrak{z} := X^\circ_\mathfrak{z} \sqcup \coprod_{x \in \mathfrak{z} \setminus \mathfrak{z}^\circ} \tilde{\mathbb{O}}_x,$$

this is an open subvariety in $X_\mathfrak{z}$ (exercise). Set $Y^1_\mathfrak{z} := X^1_\mathfrak{z} \times_{X_\mathfrak{z}} Y_\mathfrak{z}$ and define $Y'^1_\mathfrak{z}$ similarly. By construction, the morphisms $Y_z^1, Y_z'^1 \to X_z^1$ are bijective. Note that any bijective morphism to a normal variety is an iso (an exercise on the Stein decomposition), and X_z^1 is normal (b/c X_z is). So $Y_z^1 \xrightarrow{\sim} X_z^1 \xleftarrow{\sim} Y_z'^1$ extending $Y_z^0 \xrightarrow{\sim} X_z^0 \xleftarrow{\sim} Y_z'^0$.

Note that $\operatorname{codim}_{Y_z} Y_z \setminus Y_z^1, \operatorname{codim}_{Y_z'} Y_z' \setminus Y_z'^1 \geqslant 2$. We need codim $\geqslant 3$. To extend $Y_z^1 \xrightarrow{\sim} Y_z'^1$ we play a trick we used a couple of times before based on the valuative criterium of properness – but now we apply this fiberwise. Namely, the embedding $Y_z^1 \xrightarrow{\sim} Y_z'^1 \hookrightarrow Y_z'$ restricts to $Y_x^1 (:= Y_x \cap Y_x^1) \xrightarrow{\sim} Y_x'^1 \hookrightarrow Y_x'$. $\forall x$. The domain of definition of the rational map $Y_z \dashrightarrow Y_z'$ (to be denoted by Y_z^2) therefore must intersect each Y_x. So $Y_x \cap Y_z^2$ is the domain of definition of $Y_x \dashrightarrow Y_x'$. From the valuative criterium follows that $\operatorname{codim}_{Y_x} Y_x \setminus Y_z^2 \geqslant 2 \ \forall x$. And $Y_x^0 \subset Y_x^2$ so $\operatorname{codim}_{Y_z} Y_z \setminus Y_z^2 \geqslant 3$.

Since morphisms $Y_z^2 \to Y_z'$ & $Y_z'^2 \to Y_z$ extend mutually inverse isomorphisms $Y_z^0 \leftrightarrows Y_z'^0$, they restrict to mutually inverse isomorphisms $Y_z^2 \leftrightarrows Y_z'^2$, finishing the proof. $\square$

Proposition 1 can be used to relate the quantizations of $Y_\mathfrak{z}, Y'_\mathfrak{z}$. Let's start w. $\mathcal{D}_\mathfrak{z}$ on $Y_\mathfrak{z}$, and let $\mathcal{D}_{\mathfrak{z},\hbar}$ be the corresponding graded formal quantization of $Y_\mathfrak{z}$ (Sec 1.4 of Lec 22). Let $\iota: Y_\mathfrak{z}^2 \hookrightarrow Y_\mathfrak{z}$, $\iota': Y_\mathfrak{z}'^2 \hookrightarrow Y'_\mathfrak{z}$ be the inclusions and $\eta: Y_\mathfrak{z}^2 \xrightarrow{\sim} Y_\mathfrak{z}'^2$ be the isomorphism. We note that $\eta_* \iota^* \mathcal{D}_{\mathfrak{z},\hbar}$ is a graded formal quantization of $Y_\mathfrak{z}'^2$. Similarly to Thm 1.6 in Lec 22, we have $\iota'_*(\eta_*\iota^*\mathcal{D}_{\mathfrak{z},\hbar})$ is a graded formal quantization of $Y'_\mathfrak{z}$ (it's here that we use the codim $\geqslant 3$). Denote the corresponding filtered quantization by ${}'\mathcal{D}_{\mathfrak{z},\hbar}$.

Here's the 2nd key result ([BPW], Proposition 3.8).

Prop 2: We have a $\mathbb{C}[\mathfrak{z}]$-linear filtered quantization isomorphism ${}'\mathcal{D}_\mathfrak{z} \simeq \mathcal{D}'_\mathfrak{z}$.

Sketch of proof: We need to prove ${}'\mathcal{D}_{\mathfrak{z},\hbar} \simeq \mathcal{D}'_{\mathfrak{z},\hbar}$, an iso of graded formal quantizations. We are going to apply a result similar to Thm 2 in Sec 1.5 of Lec 22, more precisely a family version.

Consider the "relative regular locus" $Y_\mathfrak{z}^{rr} \subset Y_\mathfrak{z}$, where $Y_\mathfrak{z} \to \mathfrak{z}$ is smooth = the union of regular loci in Y_χ for $\chi \in \mathfrak{z}$. We have $\operatorname{codim}_{Y_\mathfrak{z}} Y_\mathfrak{z} \setminus Y_\mathfrak{z}^{rr} \geqslant 4$ (b/c $\operatorname{codim}_{Y_\chi} Y_\chi^{sing} \geqslant 4$, $\forall \chi$). From here we deduce

$H^i(Y_{\mathfrak{z}}^{rr}, \mathcal{O}) = 0$ for $i=1,2$.

One can talk about graded formal quantizations of $Y_{\mathfrak{z}}^{rr}$ (or $Y_{\mathfrak{z}}^{2,rr}$). In addition, we require that these are sheaves of $\mathbb{C}[\mathfrak{z}]$-algebras w. $\deg \mathfrak{z}^* = 2$. Thx to the cohomology vanishing condition there's a unique canonical quantization, [BK], Thm 1.8. Denote it by $\mathcal{D}_{\mathfrak{z},\hbar}^{can}$.

By [L4], Cor 2.3.2, it's uniquely characterized by the property that there is an anti-automorphism of $\mathcal{D}_{\mathfrak{z},\hbar}^{can}$, σ that is the identity mod $\hbar$, and sends $\hbar \mapsto -\hbar$.

Fact: $\mathcal{D}_{\mathfrak{z},\hbar}|_{Y^{rr}}$ is the canonical quantization (one can construct σ - essentially done in [L4], Sec 5)

So both $\mathcal{D}_{\mathfrak{z},\hbar}, \mathcal{D}'_{\mathfrak{z},\hbar}$ come w. such an anti-automorphism ($\mathcal{D}_{\mathfrak{z},\hbar}$ is the pushforward of $\mathcal{D}_{\mathfrak{z},\hbar}|_{Y^{rr}}$). Once we know that so does $\mathcal{D}'_{\mathfrak{z},\hbar}$, this will finish the proof. The check is a direct consequence of the construction of $\mathcal{D}'_{\mathfrak{z},\hbar}$ and is left as an exercise. $\square$

The following corollary establishes the independence of P.

Corollary: $\Gamma(\mathcal{D}_\mathfrak{z}) \simeq \Gamma(\mathcal{D}'_\mathfrak{z})$, a G-equivariant $\mathbb{C}[\mathfrak{z}]$-linear filtered algebra isomorphism intertwining quantum comoment maps from $\mathcal{U}(\mathfrak{g})$.

Proof: First, we show that $\Gamma(\acute{\mathcal{D}}_{\mathfrak{z},\hbar}) \simeq \Gamma(\mathcal{D}_{\mathfrak{z},\hbar})$, a $\mathbb{C}^\times$-equivariant $\mathbb{C}[\mathfrak{z}][[\hbar]]$-linear isomorphism. This follows from the construction of $\acute{\mathcal{D}}_{\mathfrak{z},\hbar} = \iota'_* \eta_* \iota^* \mathcal{D}_{\mathfrak{z},\hbar}$ & the following general fact ("quantum Hartogs").

Exercise: Let Y be a normal Poisson variety & Y° an open subset w. $\operatorname{codim}_Y Y \setminus Y^\circ \geq 2$. Let $\mathcal{D}_\hbar$ be a formal quantization of Y. Then $\Gamma(\mathcal{D}_\hbar) \xrightarrow{\sim} \Gamma(\mathcal{D}_\hbar|_{Y^\circ})$.

Note that by Proposition we have $\acute{\mathcal{D}}_{\mathfrak{z},\hbar} \simeq \mathcal{D}'_{\mathfrak{z},\hbar}$, a $\mathbb{C}[\mathfrak{z}][[\hbar]]$ linear & $\mathbb{C}^\times$-equivariant isomorphism. So $\Gamma(\mathcal{D}_{\mathfrak{z},\hbar}) \simeq \Gamma(\mathcal{D}'_{\mathfrak{z},\hbar})$. We recover $\Gamma(\mathcal{D}_\mathfrak{z})$ from $\Gamma(\mathcal{D}_{\mathfrak{z},\hbar})$ as follows (compare HW1): take $\Gamma(\mathcal{D}_{\mathfrak{z},\hbar})_{fin}$, the locally finite part for $\mathbb{C}^\times \curvearrowright \Gamma(\mathcal{D}_{\mathfrak{z},\hbar})$, then $\Gamma(\mathcal{D}_\mathfrak{z}) \simeq \Gamma(\mathcal{D}_{\mathfrak{z},\hbar})_{fin}/(\hbar-1)$. So $\Gamma(\mathcal{D}_{\mathfrak{z},\hbar}) \simeq \Gamma(\mathcal{D}'_{\mathfrak{z},\hbar})$ $\Rightarrow \Gamma(\mathcal{D}_\mathfrak{z}) \simeq \Gamma(\mathcal{D}'_\mathfrak{z})$. This isomorphism can be made G-equivariant & intertwining quantum comoment maps similarly to what was discussed in (II) of Lec 23. $\square$

1.2) 2) of the theorem.

Follows by using the same three steps as the construction of $\tilde{W}_X \curvearrowright X_\mathfrak{z}$: we have an isomorphism $\mathcal{D}_\mathfrak{z} \xrightarrow{\sim} {}^n\mathcal{D}_\mathfrak{z}$ similarly to Step 1 in Lec 24. Combining this w. Sec 1.1 (a quantum analog of Step 2 in Lec 24), for each $(n,\zeta) \in N_G(L,\tilde{O}_L)$ we get a filtered algebra automorphism $(n,\zeta): \Gamma(\mathcal{D}_\mathfrak{z}) \circlearrowleft$ that, by the construction satisfies properties (a), (b), (c). We still need to check that these automorphisms give an action of $\tilde{W}_X$ (an analog of Step 3 in that lecture). By (a), this is true on $\mathrm{gr}\,\Gamma(\mathcal{D}_\mathfrak{z})$. So the automorphisms (n,ζ) are a part of an action of an extension of $\tilde{W}_X$ by the group of G-equivariant $\mathbb{C}[\mathfrak{z}]$-linear filtered quantization automorphisms of $\Gamma(\mathcal{D}_\mathfrak{z})$. This group is unipotent (exercise). So $\tilde{W}_X$ splits giving us a required action. This finishes the proof of the theorem.

2) What would be covered if this were a 2 semester class?

We have concentrated on the case of affinizations of equivariant covers of nilpotent orbits, an example of conical symplectic singularities. This is because:

- this is most relevant (together w. slices that we didn't have time to cover) for Lie theoretic applications.
- the general case (e.g. the classification of Poisson deformations/quantizations) behaves very similarly but is less explicit.

There are other examples: e.g. symplectic quotient singularities (briefly discussed) and Hamiltonian reductions such as Nakajima quiver varieties or hypertoric varieties.

On the other hand some aspects of the structure theory such as the construction of $\mathfrak{h}_X, W_X$ from X were not discussed. Neither were Harish-Chandra bimodules (or more general HC modules). Some discussion of these topics can be found in I.L.'s MIT lectures.