# MATH 720, Lecture 3.

1) Transitive Hamiltonian actions.

2) Deformation quantization.

Refs: [CdS], Sec 22; [CG], Sec 1.3

1) In Lec 2, we have introduced symplectic forms on coadjoint orbits making them into symplectic manifolds w. transitive Hamiltonian actions. In this section we will describe all transitive Hamiltonian actions.

## 1.1) Covers

Here we introduce a slightly more general class of symplectic manifolds with transitive Hamiltonian actions: the equivariant covers of coadjoint orbits.

Let $G$ be a Lie group & $H$ its Lie subgroup. So $G/H$ is a manifold w. a $G$-action.

**Def'n:** By a $G$-equivariant cover of $G/H$ we mean the homogeneous space $G/H'$, where $H' \subset H$ is a Lie subgroup s.t.

$H/H'$ is discrete.

Rem: Let $G$ be simply connected. We write $H^\circ$ for the connected component of $1$ in $H$. Then $\pi_1(G/H) \simeq H/H^\circ$ & $G/H^\circ$ is the universal cover of $G/H$ (in the sense of Topology).

Let $\pi: G/H' \twoheadrightarrow G/H$ be the projection. Then $d_x\pi$ is iso $\forall x \in G/H'$. Assume $G/H = G\alpha \subset \mathfrak{g}^*$. Then $\pi^*\omega_{KK}$ is a $G$-invariant symplectic form on $G/H$ & the composition $G/H' \overset{\pi}{\twoheadrightarrow} G/H \hookrightarrow \mathfrak{g}^*$ is a moment map (exercise).

## 1.2) Classification of transitive Hamiltonian actions.

Theorem: Let $M$ be a Poisson manifold with a transitive Hamiltonian action of a Lie group $G$. Then the image of the moment map $\mu: M \to \mathfrak{g}^*$ is a single orbit & $\mu: M \to \operatorname{im}\mu$ is an equivariant cover (w. symplectic form lifted from $\operatorname{im}\mu$).

Sketch of proof:

Step 1: Show $M$ is symplectic $\Leftrightarrow \langle P_m, \cdot \rangle: T^*_{M,m} \xrightarrow{\sim} T_{M,m}\ \forall m \in M$. But $\dim T^*_{M,m} = \dim T_{M,m}$ so $\xrightarrow{\sim} \Leftrightarrow \twoheadrightarrow$. The surjectivity holds because $\langle \mathcal{P}_m, d_m H_\xi \rangle = \xi_{\mathcal{M},m}$ (defining condition of a comoment map) & $T_{\mathcal{M},m} = \mathrm{Span}_{\mathbb{R}}(\xi_{\mathcal{M},m} \mid \xi \in \mathfrak{g})$ (from transitivity).

Step 2: We need the following properties of $d_m\mu: T_{\mathcal{M},m} \to \mathfrak{g}^*$

Exercise: Assume $\mathcal{M}$ is general symplectic w. form $\omega$. Then:

1) For $u \in T_m\mathcal{M}$, $\langle d_m\mu(u), \xi \rangle = \omega_m(\xi_{\mathcal{M},m}, u)$
2) $\mathrm{im}\, d_m\mu = (\mathfrak{g}/\mathfrak{g}_m)^* (\subset \mathfrak{g}^*)$, $\mathfrak{g}_m = \{\xi \in \mathfrak{g} \mid \xi_{\mathcal{M},m} = 0\}$
3) $\ker d_m\mu = T_m(G_m)^{\perp\omega}$ (skew-orthogonal complement).

Back to the transitive case. Using 3) & $G_m = G$, we see that $d_m\mu: T_m\mathcal{M} \hookrightarrow \mathfrak{g}^*$. Also by the transitivity, $\mathrm{im}\,\mu$ is a single orbit. To finish the proof is left as an exercise. □

## 1.3) Orbit method as Correspondence principle.

The correspondence principle states that:

- sending $\hbar \to 0$ in a Quantum mechanical system, we get a Classical mechanical system.
- Conversely, we should be able to "quantize" (some) Classical mechanical systems to get Quantum ones.

One could also wish (not true, in general) that this preserves symmetries. And then this naturally leads to a speculation that "most symmetric" classical systems (the Hamiltonian action on $M$ is transitive) should be in some kind of relationship (ideally, a bijection) w. "most symmetric" quantum systems (where the representation is irreducible). For a simply connected nilpotent Lie group the former only includes coadjoint orbits: the stabilizers are connected (you can try to prove this). And so we get the Orbit method.

For s/simple groups, there are nontrivial covers - we'll see this in a subsequent lecture. And, as we'll hopefully see later, they play an important role in the theory.

## 1.4) Algebraic setup.

We are going to work with varieties instead of manifolds (and w. algebraic groups instead of Lie groups). If $G$ is an algebraic group, then every orbit of its algebraic action is a locally closed subvariety. Moreover, for an algebraic subgroup $H \subset G$, the group $H/H^\circ$ is finite and every equivariant cover is a variety. Coadjoint orbits & their covers are symplectic smooth varieties. The complete analog of the theorem in Sec 1.2 holds.

## 2) Deformation quantization.

Quantum Mechanics is "hard" while Classical Mechanics is "easy". So taking quasi-classical limit (converting Quantum to Classical) should be easy. But this is not the case: it's not clear how to pass from a Hilbert space to a Poisson manifold. Deformation quantization pioneered by Bayen, Flato, Fronsdal, Lichnerowitz & Sternheimer fixes this by using a more algebraic (but also more artificial) setup on the Quantum side. Taking quasi-classical limit is straightforward, while quantization becomes a problem in Deformation theory. In this class, we'll essentially use the deformation formalism.

### 2.1) Definitions.

Let $\hbar$ be an indeterminate & $\mathcal{A}_\hbar$ be a $\mathbb{C}[\hbar]$-algebra, associative & with unit. Assume

(1) $[a,b]\in \hbar \mathcal{A}_\hbar \ \forall a,b\in \mathcal{A}_\hbar$.

(2) $\hbar$ is not a zero divisor in $\mathcal{A}_\hbar$.

By (1), $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ is commutative and one can introduce a Poisson bracket on $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ by

$$\{a+\hbar\mathcal{A}_\hbar,\ b+\hbar\mathcal{A}_\hbar\}:=\underbrace{\frac{1}{\hbar}[a,b]}_{\text{exists by (1), unique by (2)}}+\hbar\mathcal{A}_\hbar$$

Exercise: Show that $\{\cdot,\cdot\}$ is well-defined & is a Poisson bracket.

Definition: Let $A$ be a Poisson algebra. By its deformation quantization we mean a pair $(\mathcal{A}_\hbar,\iota)$, where

- $\mathcal{A}_\hbar$ is as above & moreover, is complete & separated in the $\hbar$-adic topology (i.e. the natural homomorphism
$$\mathcal{A}_\hbar \to \varprojlim_n \mathcal{A}_\hbar/\hbar^n\mathcal{A}_\hbar$$
is an isomorphism).
- $\iota$ is a Poisson algebra isomorphism $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \xrightarrow{\sim} A$.

Rem: Often, we need to use a more general definition. Suppose that $\exists\, d\in \mathbb{Z}_{>0}$ s.t. $[a,b]\in\hbar^d\mathcal{A}_\hbar\ \forall a,b\in\mathcal{A}_\hbar$.

Then we can define $\{\cdot,\cdot\}$ on $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ using $\frac{1}{\hbar^\alpha}[\cdot,\cdot]$. The definition of a deformation quantization extends to this more general setting.

Most deformation quantizations we are going to consider in this course arise from "filtered quantizations."

**Definition:** Let $A$ be a Poisson algebra that, in addition, is equipped with an algebra grading by $\mathbb{Z}_{\geq 0}$: $A=\bigoplus_{i=0}^{\infty} A_i$ s.t. $\deg\{\cdot,\cdot\}=-d$ for some $d\in\mathbb{Z}_{\geq 0}$, i.e. $\{A_i,A_j\}\subset A_{i+j-d}$ $\forall i,j$. By a filtered quantization of $A$ we mean a pair $(\mathcal{A},\iota)$, where:

- $\mathcal{A}$ is an associative unital algebra equipped w. an ascending algebra filtration by $\mathbb{Z}_{\geq 0}$: $\mathcal{A}=\bigcup_{i\geq 0}\mathcal{A}_{\leq i}$ (w. $\mathcal{A}_{\leq i}\mathcal{A}_{\leq j}\subset\mathcal{A}_{\leq i+j}$). Moreover, we assume that $\deg[\cdot,\cdot]\leq -d$ (i.e. $[\mathcal{A}_{\leq i},\mathcal{A}_{\leq j}]\subset\mathcal{A}_{\leq i+j-d}$ $\forall i,j$). In this case the associated graded algebra $\operatorname{gr}\mathcal{A}=\bigoplus_{i\geq 0}\mathcal{A}_{\leq i}/\mathcal{A}_{\leq i-1}$ acquires a degree $-d$ Poisson bracket:

$$\{a+\mathcal{A}_{\leq i-1},\ b+\mathcal{A}_{\leq j-1}\}:=[a,b]+\mathcal{A}_{\leq i+j-d-1}$$

(check that this is indeed a Poisson bracket, exercise).

- $\iota: \operatorname{gr}\mathcal{A} \xrightarrow{\sim} A$ is a graded Poisson algebra isomorphism.

In the next lecture we'll explain how to get formal quantizations from filtered ones.

## 2.2) Examples of filtered quantizations.

1) Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Take $A = S(\mathfrak{g})$. The default grading satisfies the conditions of the definition w. $d=1$. Consider the universal enveloping algebra

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/(x\otimes y - y\otimes x - [x,y] \mid x,y\in\mathfrak{g}),$$

its universal property is that we have a natural Lie algebra homomorphism $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ and any Lie algebra homomorphism from $\mathfrak{g}$ to an associative algebra $\mathcal{B}$ uniquely factors through $\mathcal{U}(\mathfrak{g})$ (in particular, a representation of $\mathfrak{g}$ is the same thing as a representation of $\mathcal{U}(\mathfrak{g})$). The algebra $\mathcal{U}(\mathfrak{g})$ is filtered by the degree in $\mathfrak{g}$: $\mathcal{U}(\mathfrak{g})_{\leq k} := \operatorname{Span}_{\mathbb{C}}(x_1 \dots x_\ell \mid x_i \in \mathfrak{g} \;\&\; \ell \leq k)$.

Note that we have a natural homomorphism of graded algebras $S(\mathfrak{g}) \longrightarrow \mathrm{gr}\, U(\mathfrak{g})$ determined by $x \mapsto x + U(\mathfrak{g})_{\leq 0}$ for $x \in \mathfrak{g}$. It's a homomorphism of Poisson algebras (left as exercise). By the PBW (Poincare-Birkhoff-Witt) theorem, it's an isomorphism. So $U(\mathfrak{g})$ is a filtered quantization of $S(\mathfrak{g})$.