# Lecture 4

1) Filtered & deformation quantizations, cont'd.
2) Algebraic orbit method.

Refs: [CG], Section 1.3; [LMBM]

## 1.1) Further examples.

1 - differential operators: Let $X_0$ be a smooth affine variety, $X=T^*X_0$ so that $\mathbb{C}[X]=S_{\mathbb{C}[X_0]}(\mathrm{Vect}(X_0))$. This algebra is graded (w.r.t. degree in $\mathrm{Vect}(X_0)$) and is Poisson, Sec 1.1 in Lec 2. Recall that the bracket on $\mathbb{C}[X]$ is recovered from $\{f_1,f_2\}=0$, $\{\xi_1,f_1\}=\xi_1.f_1$, $\{\xi_1,\xi_2\}=[\xi_1,\xi_2]$ $(f_1,f_2\in\mathbb{C}[X_0], \xi_1,\xi_2\in \mathrm{Vect}(X_0))$, in particular, the degree is $-1$ $(d=1)$.

Define the algebra of (linear, algebraic) differential operators $\mathcal{D}(X_0)$ as the quotient of $T(\mathbb{C}[X_0]\oplus\mathrm{Vect}(X_0))$ by the following relations

- $f_1f_2$ is as in $\mathbb{C}[X_0]$, $\forall f_1,f_2\in\mathbb{C}[X_0]$
- $f_1\xi_1$ is as in $\mathrm{Vect}(X_0)$
- $[\xi_1,f_1]=\xi_1.f_1$

$\forall f_1\in\mathbb{C}[X_0], \xi_1\in\mathrm{Vect}(X_0)$

these hold in $\mathbb{C}[X]$ as well.

- $[\xi_1,\xi_2]$ is as in $\mathrm{Vect}(X_0)$, $\forall \xi_1,\xi_2\in\mathrm{Vect}(X_0)$

It's filtered by the degree in $\mathrm{Vect}(X_0)$. Similarly to Sec. 2.2 of Lec 3, we have a graded Poisson algebra epimorphism $\mathbb{C}[X] = S_{\mathbb{C}[X_0]}(\mathrm{Vect}(X_0)) \longrightarrow \mathrm{gr}\, \mathcal{D}(X_0)$ $(f \mapsto f, \xi \mapsto \xi + \mathcal{D}(X_0)_{\leq 0})$

It's an isomorphism (see Ex. 1.3.6 in [CG]). In particular, $\mathbb{C}[X_0] \xrightarrow{\sim} \mathcal{D}(X_0)_{\leq 0}$ & $\mathrm{Vect}(X_0) \hookrightarrow \mathcal{D}(X_0)_{\leq 1}$.

So $\mathcal{D}(X_0)$ is a filtered quantization of $\mathbb{C}[X]$.

Rem: $\mathcal{D}(X_0)$ has a natural representation in $\mathbb{C}[X_0]$, where $f \in \mathbb{C}[X_0] (\subset \mathcal{D}(X_0))$ & $\xi \in \mathrm{Vect}(X_0)$ act by the multiplication by $f$ & applying $\xi$. This representation is faithful and so elements of $\mathcal{D}(X_0)$ can be viewed as (differential) operators on $\mathbb{C}[X_0]$. Hence the name of the algebra.

2 - the Weyl algebra. Let $V$ be a finite dimensional symplectic vector space (w. form $\omega$). Consider the symmetric algebra $S(V)$, it's graded and has the unique Poisson bracket w. $\{u,v\} = \omega(u,v)$ (compare to Example 1 in Sec 1.1 of Lec 2). The degree is $-2$. It's quantization is given by the Weyl algebra $W(V) := T(V)/(u\otimes v - v \otimes u - \omega(u,v) \mid u, v \in V)$.

**Exercise:** 1) As an algebra, $W(V) \simeq \mathcal{D}(L)$, where $L \subset V$ is a Lagrangian subspace (but the filtrations are different!)

2) From here & Example 1, show that if $x_1, \ldots x_n, y_1, \ldots y_n$ is a Darboux basis of $V$ ($\omega(x_i, x_j) = \omega(y_i, y_j) = 0$, $\omega(y_i, x_j) = \delta_{ij}$) then the ordered monomials $x_1^{d_1} \ldots x_n^{d_n} y_1^{e_1} \ldots y_n^{e_n}$ form a basis in $W(V)$.

3) Deduce that $W(V)$ is a filtered quantization of $S(V)$.

## 1.2) From filtered quantizations to formal ones.

Let $\mathcal{A} = \bigcup\limits_{i \geqslant 0} \mathcal{A}_{\leq i}$ be a filtered associative algebra.

**Definition:** The Rees algebra $R_\hbar(\mathcal{A})$ is $\bigoplus\limits_{i \geqslant 0} \mathcal{A}_{\leq i} \hbar^i \subset \mathcal{A}[\hbar]$

**Exercise** : 1) Show that $R_\hbar(\mathcal{A})$ is a graded $\mathbb{C}[\hbar]$-subalgebra in $\mathcal{A}[\hbar]$ (where the grading is by degree in $\hbar$).

2) Identify $R_\hbar(\mathcal{A})/(\hbar - z)$ w. $\mathcal{A}$, $\forall z \in \mathbb{C} \setminus \{0\}$.

3) Identify $R_\hbar(\mathcal{A})/(\hbar)$ w. $\operatorname{gr} \mathcal{A}$.

We can also consider the $\hbar$-adic completion

$$\hat{R}_\hbar(\mathcal{A}) = \varprojlim R_\hbar(\mathcal{A}) / \hbar^n R_\hbar(\mathcal{A}).$$

**Proposition:** Let $A$ be a graded Poisson algebra w. $\deg\{\cdot,\cdot\}=-d$. Let $\mathcal{A}$ be a filtered quantization of $A$. Then $\hat{R}_\hbar(\mathcal{A})$ is a deformation quantization (another name: formal quantization) of $A$ (in the version, where we induce the bracket from $\frac{1}{\hbar^d}[\cdot,\cdot]$).

Sketch of proof: we have $\hat{R}_\hbar(\mathcal{A})/(\hbar) \xrightarrow[(1)]{\sim} R_\hbar(\mathcal{A})/(\hbar) \xrightarrow[(2)]{\sim} \operatorname{gr}\mathcal{A} \xrightarrow[(3)]{\sim} A$. Here (1) follows easily from the construction of $\hat{R}_\hbar(\mathcal{A})$, (2) is a part of Exer, and (3) is a part of the definition of a filtered quantization.

We claim that for $a,b\in \hat{R}_\hbar(\mathcal{A})$ we have $[a,b]\in \hbar^d \hat{R}_\hbar(\mathcal{A})$. Note that $\bigcap_{i\geq 0} \hbar^i R_\hbar(\mathcal{A})=\{0\} \Rightarrow R_\hbar(\mathcal{A})\hookrightarrow \hat{R}_\hbar(\mathcal{A})$. Also, $\hat{R}_\hbar(\mathcal{A})= R_\hbar(\mathcal{A})+\hbar^d\hat{R}_\hbar(\mathcal{A})$. So it's enough to show $[a,b]\in \hbar^d R_\hbar(\mathcal{A})$, $\forall a,b\in R_\hbar(\mathcal{A})$. It suffices to check the latter for homogeneous elements $a=\hbar^i\alpha$, $b=\hbar^j\beta$, $\alpha\in\mathcal{A}_{\leq i}$, $\beta\in\mathcal{A}_{\leq j}$. But $[\alpha,\beta]\in\mathcal{A}_{\leq i+j-d}$ and so $[a,b]=\hbar^{i+j}[\alpha,\beta]=\hbar^d(\hbar^{i+j-d}[\alpha,\beta])\in \hbar^d \underbrace{R_\hbar(\mathcal{A})_{i+j-d}}$ ← graded component

The same computation shows that the brackets on $A$ induced from $[\cdot,\cdot]$ on $\mathcal{A}$ & $\frac{1}{\hbar^d}[\cdot,\cdot]$ on $\hat{R}_\hbar(\mathcal{A})$ coincide (exercise). □

Rem: One can also pass from formal quantizations equipped with a suitably understood "grading" to filtered ones. Want to know how? Solve a homework!

## 2) Algebraic Orbit method.

In what follows, $G$ is a complex s/simple algebraic group and $\mathfrak{g}$ is its Lie algebra. Using the Killing form $(\cdot,\cdot)$ on $\mathfrak{g}$, we get a $G$-equivariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$. So all adjoint orbits & their equivariant covers are symplectic varieties.

The study of the action of $G$ on $\mathfrak{g}$ (including the orbits) is important for several reasons:

- This action has very good properties - essentially as good as one can expect (we will touch upon them). Many actions with these good properties are related to the adjoint actions of s/simple groups (e.g. Vinberg's "$\theta$-groups"). This is studied in Invariant theory. A book of Vinberg & Popov is a great survey.
- The action plays an important role in virtually all aspects of the geometric Representation theory. What is closest to this course is the representation theory of $\mathcal{U}(\mathfrak{g})$ in $\mathcal{O}$ & positive

char'c, but there are also Springer theory, the study of Hecke algebras, of representations of finite groups of Lie type.

## 2.1) Nilpotent orbits.

There is an especially important class of adjoint orbits - nilpotent ones.

**Definition**: An element $x\in\mathfrak{g}$ is *nilpotent* if the following equivalent conditions hold:

- $\underbrace{\exists \text{ faithful}}_{\forall}$ representation $\varphi:\mathfrak{g}\to End(V)$, $\varphi(x)$ is nilpotent.
- $\forall$ - - - - - - - - - - - - - -

We'll comment on the proof of the equivalence in the next lecture.

**Example**: Let $\mathfrak{g}$ be a classical Lie algebra ($\mathfrak{sl}_n, \mathfrak{so}_n$ w. $n\geqslant 3$ or $\mathfrak{sp}_n$ w. even $n$). Then $x\in\mathfrak{g}$ is nilpotent iff it's a nilpotent matrix.

Exercise: if $x$ is nilpotent, then every element in its $G$-orbit is (hint: what's a connection between representations of $\mathfrak{g}$ & of $G$?)

## 2.2) Regular functions on the orbits/covers.

Recall that every equivariant cover, $\tilde{\mathbb{O}}$, of a (co)adjoint orbit is a symplectic variety. The Poisson bivector on $\tilde{\mathbb{O}}$ gives rise to a Poisson bracket on the algebra of regular (a.k.a. polynomial) functions, $\mathbb{C}[\tilde{\mathbb{O}}]$. Also $G$ acts on $\mathbb{C}[\tilde{\mathbb{O}}]$ by algebre automorphisms. The action is rational meaning that every $f \in \mathbb{C}[\tilde{\mathbb{O}}]$ lies in a finite dimensional algebraic representation (this applies to an algebraic action of $G$ on any variety $X$, not just $\tilde{\mathbb{O}}$, - you could try to prove this).

Here are some facts that we'll elaborate on later in the course:

Fact 1: $\mathbb{C}[\tilde{\mathbb{O}}]$ is finitely generated. Moreover, the natural morphism $\tilde{\mathbb{O}} \longrightarrow X := \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$ embeds $\tilde{\mathbb{O}}$ as the unique open $G$-orbit.

Fact 2: If $\tilde{\mathbb{O}}$ is a cover of a nilpotent orbit, then $\mathbb{C}[\tilde{\mathbb{O}}]$ carries a $\mathbb{Z}_{\geqslant 0}$-grading s.t. $\deg\{\cdot,\cdot\}=-d$ for suitable $d\in\mathbb{Z}_{>0}$ (in fact, if $\tilde{\mathbb{O}}$ is an adjoint orbit, then we can take $d=1$, in general we can take $d=2$).

## 2.3) Main result.

Thx to Fact 2, it makes sense to speak about filtered quantizations of the algebras $\mathbb{C}[\tilde{\mathbb{O}}]$.

Notation: $\underline{Q}(\tilde{\mathbb{O}})$ = the set of filtered quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$, up to filtered algebra isomorphism.

Thm ("Algebraic Orbit method", LMBM):
There's a natural bijection between the two sets:
(a) $\bigsqcup_{\tilde{\mathbb{O}}} \underline{Q}(\tilde{\mathbb{O}})$, where the union is taken over all $G$-equivariant covers of all nilpotent orbits
(b) all $G$-equivariant covers of all (co)adjoint $G$-orbits.

This makes precise and proves a conjecture of Vogan from 1990.

Rem: Let's explain some ideas behind the statement & the proof. Recall that a quantization of $\mathbb{C}[\tilde{\mathbb{O}}]$ is a pair $(\mathcal{A}, \iota)$ w. $\iota: \operatorname{gr}\mathcal{A} \xrightarrow{\sim} A$. So, an isomorphism of two quantizations $(\mathcal{A},\iota), (\mathcal{A}',\iota')$ is a filtered algebra isomorphism s.t. $\iota' \circ \operatorname{gr}\varphi = \iota$. Denote the set of isomorphism classes of quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$ by $Q(\tilde{\mathbb{O}})$. The finite group $\operatorname{Aut}_G(\tilde{\mathbb{O}})$ of $G$-equivariant symplectomorphisms of $\tilde{\mathbb{O}}$ acts on $Q(\tilde{\mathbb{O}})$ (we'll explain why & how later) and

$$\underline{Q}(\tilde{\mathbb{O}}) \xrightarrow{\sim} Q(\tilde{\mathbb{O}})/\operatorname{Aut}_G(\tilde{\mathbb{O}}).$$

To relate $\underline{Q}(\tilde{\mathbb{O}})$ to the covers of all orbits we consider an important intermediate set. Later, we'll define "filtered Poisson deformations" of a graded Poisson algebra $A$. The set $P(\tilde{\mathbb{O}})$ of isomorphism classes of such deformations turns out to be naturally isomorphic to the same affine space. The proof of this uses some Algebraic geometry of $X = \operatorname{Spec}\mathbb{C}[\tilde{\mathbb{O}}]$ including that $X$ is "singular symplectic."

And then one identifies $P(\tilde{\mathbb{O}})/\operatorname{Aut}_G(\tilde{\mathbb{O}})$ w. (6) in the theorem finishing the proof.