Lecture 5.

1) Semisimple orbits & Jordan decomposition.
2) \( \mathfrak{sl}_2 \)-triples.

Refs: [B], Ch.1, Sec 6.3; [CM], Secs 2.1, 2.2, 3.1-3.4;

1.1) Semisimple elements

In Sec 2.1 of Lecture 4 we’ve defined the notion of a nilpotent element. Similarly, we can define the notion of a semisimple element.

**Definition:** An element \( x \in \mathfrak{g} \) is called \textit{semisimple} if \\
\( \exists \) faithful (equiv., \( \mathfrak{h} \)) representation \( \varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V) \), the operator \( \varphi(x) \) is s/simple (\( \Leftrightarrow \) diagonalizable).

As in the nilpotent case, if \( x \) is s/simple, then so is every element in \( \mathfrak{g}x \). So we can talk about s/simple \( \mathfrak{g} \)-orbits in \( \mathfrak{g} \). The classification of such orbits is uniform. Let \( \mathfrak{k} \subset \mathfrak{g} \) be a Cartan subalgebra and \( W \subset \mathfrak{g}(\mathfrak{k}) \) be the Weyl group.
Theorem (see Sec 2.2 in [CM]) Every element of \( \mathfrak{h} \) is simple. Every simple orbit in \( \mathfrak{g} \) intersects \( \mathfrak{h} \) at a single \( \mathcal{W} \)-orbit. This gives rise to a bijection between the set of simple orbits in \( \mathfrak{g} \) and the set \( \mathfrak{h}/\mathcal{W} \) of \( \mathcal{W} \)-orbits in \( \mathfrak{h} \).

Examples: 1) \( \mathfrak{g} = \mathfrak{gl}_n \). A semisimple element is a diagonalizable matrix. Every semisimple orbit is uniquely determined by the eigenvalues (of any of its elements). The collection of eigenvalues is an unordered \( n \)-tuple of numbers whose sum is 0 — exactly an element of \( \mathfrak{h}/\mathcal{W} \).

2) \( \mathfrak{g} = \mathfrak{so}_{2n+1} \). We realize \( \mathfrak{so}_{2n+1} \) as matrices skew-symmetric w.r.t. the main anti-diagonal: \((0, 1, \ldots, 0, -1, 0, \ldots, 0)\). For \( \mathfrak{h} \) we can take the subalgebra of all diagonal matrices: \( \mathfrak{h} = \text{diag}(x, \ldots, x, 0, -x, \ldots, -x) \subset \mathfrak{so}_{2n+1} \); \( \mathcal{W} = S_n \times \{\pm 1\}^n \) acting on \( \mathfrak{h} \) by "signed permutations."

Exercise 1: Let \( x \in \text{End}(\mathbb{C}^{2n+1}) \) be diagonalizable. For \( \lambda \in \mathbb{C} \), let \( \mathcal{V}_\lambda \) denote the \( \lambda \)-eigenspace of \( x \).
(1) Show that $x \in \mathfrak{so}_{2n+1} \iff V_\lambda^+ \oplus V_\lambda^- = \mathbb{C}^{2n+1} \not\ni \lambda.$

(2) Deduce Theorem in this case.

**Exercise 2:** Work out the examples of $g = \mathfrak{sp}_{2n} \& \mathfrak{so}_{2n}$ (the latter is more subtle) in a similar fashion.

### 1.2 Jordan decomposition

Theorem in Sec 1.1 classifies semisimple orbits. Our goal is to classify all orbits. It turns out that one can reduce the classification of all orbits in $G$ to the classification of nilpotent adjoint orbits for a smaller group. The first step is the so-called *Jordan decomposition*.

For the next theorem, see [B], Ch. 1, Sec 6.3.

**Theorem:** 1) Let $x \in g$. Then $\exists!$ s/simple $x_s$ & nilpotent $x_n \in g$ s.t. $[x_s, x_n] = 0 \& x_s + x_n = x$ (the *Jordan decomposition*).

2) Let $\varphi: g \rightarrow \tilde{g}$ be a homomorphism of s/simple Lie algebras. Then $\varphi(x)_s = \varphi(x_s), \varphi(x)_n = \varphi(x_n) \not\ni x \in g.$

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Exercise: \( \cdot \) Verify 1) for \( g = \mathfrak{gl}_n \).

\( \cdot \) Deduce the equivalence of conditions in the definition of nilpotent element (Sec 2.1 of Lec 4) from Thm.

1.3) Levi subgroups & reduction to classification of nilpotent orbits.

Definition: By a\textbf{ Levi subgroup of }\( G \) we mean the centralizer of a simple element in \( g \).

Examples: 1) \( G = SL_n \) Up to conjugation, a semisimple element \( x \) is \( \text{diag}(x_1, x_2, \ldots, x_k) \) w. \( x_i \neq x_j \) for \( i \neq j \). The centralizer \( Z_G(x) \) consists of the block diagonal matrices w. blocks of sizes \( m_1, \ldots, m_k \).

2) \( G = SO_{2n+1} \) Can assume \( x = \text{diag}(x_1, -x_1, x_2, -x_2, \ldots, x_n, -x_n) \).

Exercise 1: Identify \( Z_G(x) \) w. \( \prod_{i=1}^k \mathfrak{gl}(m_i) \times \mathfrak{so}(m_0) \).

Fact: For the general \( G \), every Levi subgroup \( L \) is a connected reductive group.
In particular, $(L,L)$ is a semisimple group, $L = z(L) \oplus [L,L]$, and all elements in $z(L)$ are simple.

**Exercise 2:** Check these claims in the examples.

**Proposition:** Fix a semisimple element $x \in g$. Let $L = \mathbb{Z}_g(x)$.

There's a bijection between:

1. The $G$-orbits $Gy \leq g$ w. $Gy_5 = Gx$

The map $(2) \rightarrow (1)$ sends $(L,L)y'$ to $G(x+y')$.

**Sketch of proof:** Let's construct a map $(1) \rightarrow (2)$. We can assume $y_5 = 1$. We have $(gy)_5 = x \iff (2)$ of Thm in Sec 1.2 applied to the automorphism $g$ of $g$. $g \cdot y_5 = x \iff g \in L$. We claim $y_n \in [L,L]$. Indeed, let $\pi_1, \pi_2$ denote the projections $L \rightarrow z(L), [L,L]$ so that $y_n = \pi_1(y_n) + \pi_2(y_n)$. But $\pi_1(y_n)$ is semisimple and if it's $\neq 0$, then $(y_n)_5 = \pi_1(y_n) + \pi_2(y_n)_5 \neq 0$, a contradiction w. $y_n$ being nilpotent.

Also, $L = \mathbb{Z}(L)(L,L)$ implies that each $L$-orbit in $L$ is a...
single $(L,L)$-orbit. The map $(1) \to (2)$ sends $L_y$ to $(L,L)\gamma$, where we choose $y \in \gamma$, $y_3 = x$.

**Exercise 3:** Show that the two maps are well-defined & mutually inverse.

**2) $s_L$-triples.**

Here we explain an approach to studying nilpotent orbits. We will relate them to ($L$-conjugacy classes of) homomorphisms $s_L \to g$, a.k.a. "$s_L$-triples." The point of this: we can use the representation theory of $s_L$ to study the nilpotent orbits — we will do so in this lecture & subsequent ones.

**Definition:** An $s_L$-triple in $g$ is $(e,h,f) \in g^3$ s.t. the defining relations of $s_L$ hold: $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$.

Of course, to give such is to give a homomorphism $s_L \to g$.

Note that $e$ is nilpotent: this follows, e.g., from 2) of Thm in Sec 1.2 — but can also be proved directly.
Theorem (Jacobson-Morozov: [CM], Sec 3.2)
Every nilpotent element $eefg$ is included into an $\mathfrak{sl}_2$-triple.

Theorem (Kostant) Let $(e,h,f), (e,h',f')$ be $\mathfrak{sl}_2$-triples. Then $\exists g \in G$ s.t. $g.e = e, g.h = h', g.f = f'$.

This theorem will be proved below.

Cor: The map $(e,h,f) \mapsto e$ gives rise to a bijection between:
- $G$-conjugacy of $\mathfrak{sl}_2$-triples
- Nilpotent $G$-orbits.

Proof: JM theorem says the map is surjective & Kostant’s thm says the map is injective. □

Example: $\mathfrak{sl}_2 \rightarrow \mathfrak{sl}_n$. A homomorphism $\mathfrak{sl}_2 \rightarrow \mathfrak{sl}_n$ is an $n$-dimensional $\mathfrak{sl}_2$-rep. $\mathfrak{sl}_n$-conjugacy class = isomorphism class. Recall that fin. dimensional $\mathfrak{sl}_2$-reps are completely reducible and for each dimen-
sion $\exists!$ irrep. It follows that the $n$-dimensional $\mathfrak{sl}_2$-reps are
classified by the partitions of $n$. Also in each $\mathfrak{sl}_2$-irrep in
the standard basis, $e$ acts as a single Jordan block. So
Corollary recovers the classification of nilpotent orbits in $\mathfrak{sl}_n$ via Jordan types.

2.1) Proof of Kostant's theorem

We will need a slightly stronger claim, where we choose $g$ from a certain subgroup of $Z_i = Z_i(e)$. For $i \in \mathcal{I}$, set $g_i = \{x \in g \mid [h,x] = i x \}$, $Z_i = Z_i g_i$. From the rep theory of $\mathfrak{sl}_2$, we deduce that $Z = \bigoplus_{i \in \mathcal{I}} Z_i$. Consider the ideal $Z_+ = \bigoplus_{i \in \mathcal{I}} Z_i$ in $Z$.

It is contained in $\bigoplus_{i \in \mathcal{I}} Z_i$; and the latter subalgebra consists of nilpotent elements (Exercise: check this in examples). So $Z_+$ consists of nilpotent elements and hence $Z_+ = \exp(Z_+)$ is an algebraic subgroup of $Z$.

Exercise: $Z_+$ is normal in $Z$. It's unipotent as an algebraic group.

The following claim implies Kostant's theorem.

Proposition: Let $(e,h,f)$, $(e,h',f')$ be two $\mathfrak{sl}_2$-triples. Then $\exists g \in Z_+$ w. $gh = h'$, $gf = f'$. 

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Proof: Step 1:

Claim: \( Z_+ h = h + z_+ \)

Exercise: prove this using that \( z_+ = \oplus z_i \) & \( Z_+ = \exp(z_+) \).

Step 2: Here we show that \( h' \in Z_+ h = [\text{Step 1}] = h + z_+ \iff h' - h \in z_+ \). Note that \( [e, h' - h] = -2e + 2e = 0 \) & \( h' - h = [e, f' - f] \in \text{im} \[e, \cdot\] \). From the rep. theory of \( \mathfrak{sl}_2 \), we know that \( z = \ker \[e, \cdot\] \cap \text{im} \[e, \cdot\] = z_+ \); \( h' - h \in z_+ \) follows.

Step 3: We can apply an element of \( Z_+ \) to \( (h, f) \) and assume \( h' = h \). We claim that then \( f' = f \). Indeed, \( [e, f' - f] = 0 \iff f' - f \in z \). But \( f' - f \in z \). But \( z = \{0\} \), so \( f' = f \). \( \square \)