Lecture 5.

1) Semisimple orbits & Jordan decomposition.
2) $\mathfrak{sl}_2$-triples.

Refs: [B], Ch. 1, Sec 6.3; [CM], Secs 2.1, 2.2, 3.1-3.4;

1.1) Semisimple elements

In Sec 2.1 of Lecture 4 we've defined the notion of a nilpotent element. Similarly, we can define the notion of a semisimple element.

Definition: An element $x \in \mathfrak{g}$ is called *semisimple* if $\exists$ faithful (equiv., $\forall$) representation $\varphi: \mathfrak{g} \to \mathfrak{gl}(V)$, the operator $\varphi(x)$ is s/simple ($\Leftrightarrow$ diagonalizable).

As in the nilpotent case, if x is s/simple, then so is every element in Gx. So we can talk about s/simple G-orbits in $\mathfrak{g}$. The classification of such orbits is uniform. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $W \subset GL(\mathfrak{h})$ be the Weyl group.

Theorem (see Sec 2.2 in [CM]) Every element of $\mathfrak{h}$ is s/simple. Every s/simple orbit in $\mathfrak{g}$ intersects $\mathfrak{h}$ at a single W-orbit. This gives rise to a bijection between the set of s/simple orbits in $\mathfrak{g}$ and the set $\mathfrak{h}/W$ of W-orbits in $\mathfrak{h}$.

Examples: 1) $\mathfrak{g}=\mathfrak{sl}_n$. A semisimple element is a diagonalizable matrix. Every semisimple orbit is uniquely determined by the eigenvalues (of any of its elements). The collection of eigenvalues is an unordered n-tuple of numbers whose sum is 0 – exactly an element of $\mathfrak{h}/W$.

2) $\mathfrak{g}=\mathfrak{so}_{2n+1}$. We realize $\mathfrak{so}_{2n+1}$ as matrices skew-symmetric w.r.t. the main anti-diagonal: $\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$. For $\mathfrak{h}$ we can take the subalgebra of all diagonal matrices: $\mathfrak{h}=\operatorname{diag}(x_1,\dots x_n, 0, -x_n,\dots,-x_1)\subset\mathfrak{so}_{2n+1}$; $W=S_n\ltimes\{\pm1\}^n$ acting on $\mathfrak{h}$ by "signed permutations".

Exercise 1: Let $x\in \operatorname{End}(\mathbb{C}^{2n+1})$ be diagonalizable. For $\lambda\in\mathbb{C}$, let V_λ denote the λ-eigenspace of x.

(1) Show that $x \in \mathfrak{so}_{2n+1} \iff V_\lambda^\perp \oplus V_{-\lambda} = \mathbb{C}^{2n+1}$ $\forall \lambda$.

(2) Deduce Theorem in this case.

Exercise 2: Work out the examples of $\mathfrak{g} = \mathfrak{sp}_{2n}$ & $\mathfrak{so}_{2n}$ (the latter is more subtle) in a similar fashion.

1.2) Jordan decomposition

Theorem in Sec 1.1 classifies semisimple orbits. Our goal is to classify <u>all</u> orbits. It turns out that one can reduce the classification of all orbits in G to the classification of <u>nilpotent</u> adjoint orbits for a <u>smaller</u> group. The first step is the so called Jordan decomposition.

For the next theorem, see [B], Ch. 1, Sec 6.3.

Theorem: 1) Let $x \in \mathfrak{g}$. Then $\exists!$ s/simple x_s & nilpotent $x_n \in \mathfrak{g}$ s.t. $[x_s, x_n] = 0$ & $x_s + x_n = x$ (the Jordan decomposition)

2) Let $\varphi: \mathfrak{g} \to \tilde{\mathfrak{g}}$ be a homomorphism of s/simple Lie algebras. Then $\varphi(x)_s = \varphi(x_s)$, $\varphi(x)_n = \varphi(x_n)$ $\forall x \in \mathfrak{g}$.

Exercise: • Verify 1) for $\mathfrak{g} = \mathfrak{sl}_n$.

• Deduce the equivalence of conditions in the definition of nilpotent element (Sec 2.1 of Lec 4) from Thm.

1.3) Levi subgroups & reduction to classification of nilpotent orbits.

Definition: By a *Levi subgroup* of G we mean the centralizer of a s/simple element in $\mathfrak{g}$.

Examples: 1) $G = SL_n$. Up to conjugation, a semisimple element x is $\mathrm{diag}(\underbrace{x_1,\dots x_1}_{m_1}, \underbrace{x_2,\dots x_2}_{m_2},\dots, \underbrace{x_k,\dots x_k}_{m_k})$ w. $x_i \neq x_j$ for $i \neq j$. The centralizer $Z_G(x)$ consists of the block diagonal matrices w. blocks of sizes $m_1, \dots m_k$.

2) $G = SO_{2n+1}$. Can assume $x = \mathrm{diag}(\overbrace{x_k,\dots x_k}^{m_k},\dots, \overbrace{x_1,\dots x_1}^{m_1}, \overbrace{0,\dots 0}^{m_0}, \overbrace{-x_1,\dots -x_1}^{m_1},\dots \overbrace{-x_k,\dots -x_k}^{m_k})$

Exercise 1: Identify $Z_G(x)$ w. $\prod_{i=1}^{k} GL(m_i) \times SO(m_0)$.

Fact: For the general G, every Levi subgroup L is a connected reductive group.

In particular, (L,L) is a semisimple group, $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l},\mathfrak{l}]$, and all elements in $\mathfrak{z}(\mathfrak{l})$ are s/simple.

Exercise 2: Check these claims in the examples.

Proposition: Fix a semisimple element $x \in \mathfrak{g}$. Let $L = Z_G(x)$. There's a bijection between:

(1) The G-orbits $Gy \subset \mathfrak{g}$ w. $Gy_s = Gx$

(2) The nilpotent orbits of (L,L) (in $[\mathfrak{l},\mathfrak{l}]$)

The map (2) $\to$ (1) sends $(L,L)y'$ to $G(x+y')$.

Sketch of proof: Let's construct a map (1) $\to$ (2). We can assume $y_s = x$. We have $(g.y)_s = x \Leftrightarrow$ [(2) of Thm in Sec 1.2 applied to the automorphism g of $\mathfrak{g}$] $g.y_s = x \Leftrightarrow$ $g \in L$. We claim $y_n \in [\mathfrak{l},\mathfrak{l}]$. Indeed, let π_1, π_2 denote the projections $\mathfrak{l} \twoheadrightarrow \mathfrak{z}(\mathfrak{l}), [\mathfrak{l},\mathfrak{l}]$ so that $y_n = \pi_1(y_n) + \pi_2(y_n)$. But $\pi_1(y_n)$ is semisimple and if it's $\neq 0$, then $(y_n)_s = \pi_1(y_n) + \pi_2(y_n)_s$ $\neq 0$, a contradiction w. y_n being nilpotent.

Also $L = Z(L)(L,L)$ implies that each L-orbit in $\mathfrak{l}$ is a single (L,L)-orbit. The map $(1)\to(2)$ sends G_y to $(L,L)y_n$, where we choose y w. $y_s=x$.

Exercise 3: Show that the two maps are well-defined & mutually inverse. $\square$

2) $\mathfrak{sl}_2$-triples.

Here we explain an approach to studying nilpotent orbits. We will relate them to (G-conjugacy classes of) homomorphisms $\mathfrak{sl}_2\to\mathfrak{g}$, a.k.a. "$\mathfrak{sl}_2$-triples." The point of this: we can use the representation theory of $\mathfrak{sl}_2$ to study the nilpotent orbits - we will do so in this lecture & subsequent ones.

Definition: An $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ is $(e,h,f)\in\mathfrak{g}^3$ s.t. the defining relations of $\mathfrak{sl}_2$ hold: $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$.

Of course, to give such is to give a homomorphism $\mathfrak{sl}_2\to\mathfrak{g}$.

Note that e is nilpotent: this follows, e.g., from 2) of Thm in Sec 1.2 - but can also be proved directly.

Theorem (Jacobson-Morozov: [CM], Sec 3.2)

Every nilpotent element $e \in \mathfrak{g}$ is included into an $\mathfrak{sl}_2$-triple.

Theorem (Kostant) Let (e,h,f), (e,h',f') be $\mathfrak{sl}_2$-triples. Then $\exists\, g \in G$ s.t. $g.e = e$, $g.h = h'$, $g.f = f'$.

This theorem will be proved below.

Cor: The map $(e,h,f) \mapsto e$ gives rise to a bijection between:

- G-conjugacy of $\mathfrak{sl}_2$-triples
- Nilpotent G-orbits.

Proof: JM theorem says the map is surjective & Kostant's thm says the map is injective. $\square$

Example: $\mathfrak{g} = \mathfrak{sl}_n$. A homomorphism $\mathfrak{sl}_2 \to \mathfrak{sl}_n$ is an n-dimensional $\mathfrak{sl}_2$-rep. SL_n-conjugacy class = isomorphism class. Recall that fin. dimensional $\mathfrak{sl}_2$-reps are completely reducible and for each dimension $\exists!$ irrep. It follows that the n-dimensional $\mathfrak{sl}_2$-reps are classified by the partitions of n. Also in each $\mathfrak{sl}_2$-irrep in the standard basis, e acts as a single Jordan block. So

Corollary recovers the classification of nilpotent orbits in $\mathfrak{sl}_n$ via Jordan types.

2.1) Proof of Kostant's theorem

We will need a slightly stronger claim, where we choose g from a certain subgroup of $Z := Z_G(e)$. For $i \in \mathbb{Z}$, set $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [h,x] = ix\}$, $\mathfrak{z}_i = \mathfrak{z} \cap \mathfrak{g}_i$. From the rep. theory of $\mathfrak{sl}_2$ we deduce that $\mathfrak{z} = \bigoplus_{i \geqslant 0} \mathfrak{z}_i$. Consider the ideal $\mathfrak{z}_+ = \bigoplus_{i > 0} \mathfrak{z}_i$ in $\mathfrak{z}$. It's contained in $\bigoplus_{i > 0} \mathfrak{g}_i$ and the latter subalgebra consists of nilpotent elements (exercise: check this in examples). So $\mathfrak{z}_+$ consists of nilpotent elements and hence $Z_+ := \exp(\mathfrak{z}_+)$ is an algebraic subgroup of Z.

Exercise: Z_+ is normal in Z. It's unipotent as an algebraic group.

The following claim implies Kostant's theorem.

Proposition: Let (e,h,f), (e,h',f') be two $\mathfrak{sl}_2$-triples. Then $\exists\, g \in Z_+$ w. $gh = h'$, $gf = f'$.

Proof: Step 1:

Claim: $\mathcal{Z}_+ h = h + \mathfrak{z}_+$

Exercise: prove this using that $\mathfrak{z}_+ = \bigoplus_{i>0} \mathfrak{z}_i$ & $\mathcal{Z}_+ = \exp(\mathfrak{z}_+)$.

Step 2: Here we show that $h' \in \mathcal{Z}_+ h = [\text{Step 1}] = h + \mathfrak{z}_+ \Leftrightarrow$ $h' - h \in \mathfrak{z}_+$. Note that $[e, h'-h] = -2e + 2e = 0$ & $h'-h = [e, f'-f] \in \operatorname{im}[e, \cdot]$. From the rep. theory of $\mathfrak{sl}_2$, we know that $\mathfrak{z} (= \ker [e, \cdot]) \cap \operatorname{im}[e, \cdot] = \mathfrak{z}_+$; $h' - h \in \mathfrak{z}_+$ follows.

Step 3: We can apply an element of $\mathcal{Z}_+$ to (h, f) and assume $h' = h$. We claim that then $f' = f$. Indeed, $[e, f'-f] = 0$ $\Leftrightarrow f' - f \in \mathfrak{z}$. But $f' - f \in \mathfrak{g}_{-2}$. But $\mathfrak{z}_{-2} = \{0\}$, so $f' = f$. $\square$