Lecture 6.

1) Structure of the centralizer.
2) Nilpotent orbits in classical Lie algebras.
3) Equivariant covers of nilpotent orbits

Refs: [CM], Secs 3.7, 5.1, 6.1.

1) Structure of the centralizer.

The main goal of this lecture is to give a classification of nilpotent orbits in the classical Lie algebras & describe their equivariant covers. In several parts we will need a result on the structure of the centralizer $Z_G(e)$.

In Section 2.1 of Lecture 5 we have introduced the unipotent normal subgroup $Z_+ \subset Z$. On the other hand consider the subgroup $Q := Z_G(e,h,f)$.

Example: Let $\mathfrak{g} = \mathfrak{sl}_n$ and let $\mathbb{O}$ be a nilpotent orbit corresponding to a partition τ of n, $\tau = (n_1^{\alpha_1}, \ldots n_k^{\alpha_k})$ ← mult's

Then Q is the intersection of SL_n w. the automorphism

group of the corresponding representation of $\mathfrak{sl}_2$, which is $\prod_{i=1}^{k} GL_{d_i}$ embedded via $(g_1,\dots,g_k) \mapsto \mathrm{diag}(\overbrace{g_1,\dots g_1}^{n_1},\dots \overbrace{g_k,\dots g_k}^{n_k})$.
Note that Q is reductive in this case.

Proposition: i) We have $Z_G(e) = Q \ltimes Z_+$
ii) Q is a reductive algebraic group.

We will compute the groups Q for other classical G below (after we classify the orbits). We'll see they are reductive in these cases.

Proof of Proposition: i) We need to show $Q \cap Z_+ = \{1\}$ & $QZ_+ = Z$.

- $Q \cap Z_+ = \{1\}$. Note that $\mathfrak{q} \overset{\text{in fact equal}}{\subset} \mathfrak{z} \cap \mathfrak{g}_0 = \mathfrak{z}_0$ & $\mathfrak{z}_0 \cap \mathfrak{z}_+ = \{0\}$.

So $Q \cap Z_+$ is finite. Since Z_+ is unipotent, $Q \cap Z_+ = \{1\}$.

- $QZ_+ = Z$. Take $g \in Z$. Then $(e, g.h, g.f)$ is an $\mathfrak{sl}_2$-triple

By Prop. in Sec 2.1 of Lec 5 $\exists\, g' \in Z_+$ s.t. $g'h = gh$, $g'f = gf$.
So $g'g^{-1} \in Q$. $\square$

ii – sketch: One needs to show that $\mathfrak{q}$ is a "reductive subalgebra" of $\mathfrak{g}$ (in the sense of [B], Ch. 1, Sec 6.6). One checks that the restriction of the Killing form to $\mathfrak{q}$ is nondegenerate (exercise) and deduces that $\mathfrak{q}$ is reductive from there. $\square$

Rem: Every algebraic group decomposes as the semidirect product of a normal unipotent group (a.k.a. the unipotent radical) and a reductive group. This is the so called Levi decomposition (see [OV], Section 6).

2) Nilpotent orbits in classical Lie algebras.

The goal here is to use the bijection between nilpotent orbits & conjugacy classes of $\mathfrak{sl}_2$-triples to give a combinatorial classification (essentially, in terms of partitions) of nilpotent orbits in the classical Lie algebras $\mathfrak{so}_n$ & $\mathfrak{sp}_n$.

Set $G := O_n$ or Sp_n (note that the former is disconnected w. component group $\mathbb{Z}/2\mathbb{Z}$). We start w. classifying the nilpotent G-orbits in $\mathfrak{g}$.

Definition: Let τ be a partition of n. We say that τ is
- Of O-type if every even part occurs in τ w. even mult'y
- Of Sp-type — " — odd — " —

For example, $(2,2,1,1)$ is both, $(3,2,2,1,1)$ is only of O-type, $(3,3,2,1,1)$ is only of Sp-type, $(3,2,1,1)$ is of neither type.

Theorem: The nilpotent G-orbits in $\mathfrak{g}$ are classified by the partitions of the corresponding type (via taking the Jordan type).

Sketch of proof: Step 0: For $\mathfrak{g}=\mathfrak{so}_n$ (resp. $\mathfrak{sp}_n$) a homomorphism $\mathfrak{sl}_2\to\mathfrak{g}$ is an n-dimensional representation of $\mathfrak{sl}_2$ that admits an $\mathfrak{sl}_2$-invariant orthogonal (resp. symplectic) form. Two such representations are G-conjugate iff $\exists$ orthogonal/symplectic isomorphism between them.

The proofs of steps below are left as exercises.

Step 1: Let $V(m)$ denote the m-dimensional $\mathfrak{sl}_2$-irrep. Then $V(m)$ admits a (unique up to rescaling) $\mathfrak{sl}_2$-invariant non-degenerate bilinear form that is orthogonal for m odd & symplectic for m even.

Step 2: Let U, V be two fin. dim. vector spaces each equipped with orthogonal or symplectic forms β_U, β_V. Then the form $\beta_U \otimes \beta_V$ on $U \otimes V$ is orthogonal if β_U, β_V are of the same type and symplectic else.

Step 3: Let V be an $\mathfrak{sl}_2$-rep w. invariant orthogonal/symplectic form. The space $\mathrm{Hom}_{\mathbb{C}}(V(m), V)$ acquires an orthogonal/symplectic form thx to Steps 1 & 2. Show that this form is $\mathfrak{sl}_2$-invariant and deduce that its restriction to the multiplicity space $M_m := \mathrm{Hom}_{\mathfrak{sl}_2}(V(m), V)$ is non-degenerate. In particular, if the forms on $V(m)$ & V are of different types, then the mult. space is even dimensional - which is where the restriction on partitions comes from.

Step 4: The natural isomorphism

$$\bigoplus_{m \geq 0} M_m \otimes V(m) \xrightarrow{\sim} V$$

preserves the forms, where the form on the source is $\oplus$ of the forms from Step 2. Since every (resp. every even dimensional) vector space has the unique orthogonal (resp. symplectic) form up to iso, the dimensions of multiplicity spaces determine V uniquely up to orthogonal/symplectic iso. $\square$

Since the group Sp_n is connected, the theorem gives the classification of nilpotent orbits in $\mathfrak{sp}_n$. For n odd, we have $O_n = SO_n \times Z(O_n)$ w. $Z(O_n) = \{\pm id\}$, so the SO_n & O_n orbits in $\mathfrak{so}_n$ coincide (the center acts trivially). The classification of nilpotent orbits is complete in this case as well. To understand the case of $\mathfrak{so}_n$ w. n even we need to digress and discuss the structure of $Q = Z_G(e,h,f)$.

Proposition: Let $G = O_n$ or Sp_n and $e \in \mathfrak{g}$ be a nilpotent orbit from the G-orbit labelled by a partition $\tau = (n_1^{d_1}, \dots, n_k^{d_k})$. Then:

1) If $G = O_n$, then $Q \simeq \prod_{n_i \text{ odd}} O_{d_i} \times \prod_{n_i \text{ even}} Sp_{d_i}$

2) If $G = Sp_n$, then $Q \simeq \prod_{n_i \text{ even}} O_{d_i} \times \prod_{n_i \text{ odd}} Sp_{d_i}$

In both cases, the embedding is via

$$(g_1, \dots g_k) \mapsto \operatorname{diag}(\underbrace{g_1, \dots g_1}_{n_1}, \underbrace{g_2, \dots g_2}_{n_2}, \dots, \underbrace{g_k, \dots g_k}_{n_k})$$

in a suitable basis.

Proof: Q is the group of all $g \in GL(V)$ that are

(i) $\mathfrak{sl}_2$-linear &

(ii) preserve the form on V.

Note that $\mathrm{Aut}_{\mathfrak{sl}_2}(V) \xrightarrow{\sim} \prod_m GL(M_m)$, where $M_m = \mathrm{Hom}_{\mathfrak{sl}_2}(V(m), V)$ is the multiplicity space. An element of $\mathrm{Aut}_{\mathfrak{sl}_2}(V)$ satisfies (ii) $\Leftrightarrow$ [Step 4 of proof] its image in $GL(M_m)$ preserves the form (orthogonal if the types of forms on $V(m)$ & V are the same, symplectic else). This implies the claim of Prop'n. $\square$

Now we proceed to describing nilpotent SO_n-orbits in $\mathfrak{so}_n$ for n even. Since $O_n/SO_n \simeq \mathbb{Z}/2\mathbb{Z}$, every O_n-orbit is either a single SO_n-orbit or is the disjoint union of two SO_n-orbit.

Corollary: Let $\mathbb{O}$ be the nilpotent O_n-orbit corresponding to a partition τ of n. Then $\mathbb{O}$ splits into the disjoint union of two SO_n-orbits $\Leftrightarrow$ all parts of τ are even (such τ are called very even.

Proof: Let $e \in \mathbb{O}$. We have $SO_n.e = O_n.e \Leftrightarrow Z_{SO_n}(e) \subsetneq Z_{O_n}(e)$ $\Leftrightarrow Z_{O_n}(e) \not\subset SO_n$. By Prop'n in Sec 1.1, $Z_{O_n}(e) = Z_{O_n}(e,h,f) \ltimes Z_{+}$, where Z_+ is unipotent, hence connected. So $Z_{O_n}(e) \subset SO_n$ $\Leftrightarrow Z_{O_n}(e,h,f) \subset SO_n$. Now we use 1) of Proposition: every O_{d_i}-factor has an element w. $\det = -1$. So our condition is equivalent to the claim that there's no such factor $\square$

3) Equivariant covers of nilpotent orbits

As was mentioned in Sec 1.1 of Lec 3, the G-equivariant covers of the orbit G/H are parameterized by subgroups in H/H°. So to understand the covers of $\mathbb{O} := Ge$ we need to compute $Z_G(e)/Z_G(e)^\circ$. Since Z_+ is connected, we get $Z_G(e)/Z_G(e)^\circ \xrightarrow{\sim} Q/Q^\circ$. Now we can use Example in Sec. 1 (for $G = SL_n$ & results on computing Q from Sec 2 (for $G = SO_n, Sp_n$) to describe Q/Q° for the classical groups. The proof of the main result of this section is left as an exercise.

Proposition: Let τ denote the partition corresponding to a nilpotent orbit in $\mathbb{O}$, $\tau = (n_1^{d_1}, \dots, n_k^{d_k})$

1) Let $G = SL_n$. Then $Q/Q^\circ \simeq \mathbb{Z}/GCD(n_1,\dots,n_k)\mathbb{Z}$. Moreover,

$Z(G) \twoheadrightarrow Q/Q^\circ$.

2) Let $G = Sp_n$. Then $Q/Q^\circ \simeq (\mathbb{Z}/2\mathbb{Z})^a$, where $a := \#\{i \mid n_i \text{ is even}\}$.

3) Let $G = SO_n$. Then $Q/Q^\circ \simeq (\mathbb{Z}/2\mathbb{Z})^{\max(b-1,0)}$, where $b := \{i \mid n_i \text{ is odd}\}$.

Example: $G = Sp_n$, $\tau = (2, 1^{n-2}) \rightsquigarrow \mathbb{O} \subset \mathfrak{sp}_n$. Then $Q/Q^\circ \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$. The universal cover of $\mathbb{O}$, $G/Z_G(e)^\circ$ is G-equivariantly identified with $\mathbb{C}^n \setminus \{0\}$ w. its natural G-action. The covering map $\mu : \mathbb{C}^n \setminus \{0\} \twoheadrightarrow \mathbb{O}$ is the moment map. The proof is an exercise.

Remarks: 1) $G = SO_n$ is not simply connected. Its simply connected cover is $Spin_n$. The group Q/Q° for $Spin_n$ is sometimes different from the SO_n-case and may be non-commutative. See [CM], Section 6.1.

2) The classification of nilpotent orbits in exceptional types as well as the computation of the component groups are known, see [CM], Section 8.4.