Lecture 7.

1) Finiteness of number of nilpotent orbits
2) Algebra $\mathbb{C}[\tilde{\mathbb{O}}]$

Refs: [CM], Secs 3.4, 3.5; [PV], Sec 1.5; [J], Sec 8.

1) Finiteness of number of nilpotent orbits

In the previous two lectures we have explained how to classify the nilpotent orbits in the classical Lie algebras combinatorially. In particular, we see that there are finitely many nilpotent orbits in these algebras. Turns out, this holds for arbitrary s/simple Lie algebras. In what follows we give an argument bases on [PV], Sec. 1.5. For an argument using the theory of $\mathfrak{sl}_2$-triples, [CM], Sec. 3.4 (Mal'cev's thm) & Sec 3.5.

Let $G \subset GL(V)$ be a s/simple algebraic subgroup.

Proposition: $\forall$ $GL(V)$-orbit $\mathbb{O} \subset \mathfrak{gl}(V)$, $\mathbb{O} \cap \mathfrak{g}$ is the disjoint union of finitely many G-orbits (as a scheme).

Proof: It's enough to show that, $\forall\ x\in\mathbb{O}\cap\mathfrak{g}$, we have
$$T_x\, G.x = T_x(\mathbb{O}\cap\mathfrak{g}) \qquad (1)$$
where in the r.h.s. we view $\mathbb{O}\cap\mathfrak{g}$ w. its natural scheme str're.
Indeed, $G.x$ is a locally closed subset in $\mathbb{O}\cap\mathfrak{g}$. Since it's smooth, $\dim T_x G.x = \dim Gx$. On the other hand, $\dim T_x(\mathbb{O}\cap\mathfrak{g}) \geqslant \dim_x \mathbb{O}\cap\mathfrak{g}$ (the local dim'n). (1) shows that Gx is open in $\mathbb{O}\cap\mathfrak{g}$. Since this is true $\forall x$, (1) implies the claim of proposition.

We have $T_x(Gx) = [\mathfrak{g},x]$, while $T_x(\mathbb{O}\cap\mathfrak{g}) = T_x\mathbb{O}\cap\mathfrak{g} =$
$= [\mathfrak{gl}(V),x]\cap\mathfrak{g}$. We need to show $[\mathfrak{g},x] = [\mathfrak{gl}(V),x]\cap\mathfrak{g}$.

Every rational representation of G is completely reducible. In particular, $\mathfrak{gl}(V) = \mathfrak{g}\oplus\mathfrak{g}^\circ$ as G-reps.
$$[\mathfrak{gl}(V),x] = [\mathfrak{g}\oplus\mathfrak{g}^\circ, x] = [x\in\mathfrak{g}] = \underbrace{[\mathfrak{g},x]}_{\subset\mathfrak{g}} \oplus \underbrace{[\mathfrak{g}^\circ,x]}_{\subset\mathfrak{g}^\circ}$$
So $[\mathfrak{gl}(V),x]\cap\mathfrak{g} = [\mathfrak{g},x]$, showing (1). $\square$

Corollary 1: Let $\mathfrak{g}$ be a s/simple Lie algebra. Pick s/simple $x\in\mathfrak{g}$. Then $\#\{Gy\subset\mathfrak{g} \mid Gy_s = Gx\} < \infty$. In particular, there are fin. many nilpotent orbits (case $x=0$).

Proof: Let V be a faithful rep. of some G w. $\mathrm{Lie}(G) = \mathfrak{g}$. There are finitely many $GL(V)$-orbits in $\mathfrak{gl}(V)$ whose s/simple part is in $GL(V)x$ (by JNF thm). By Proposition, each of them intersects $\mathfrak{g}$ at finitely many orbits. These are precisely the G-orbits in question. $\square$

Corollary 2: Let $\mathbb{O} \subset \mathfrak{g}$ be an orbit. Then $\overline{\mathbb{O}} \subset \mathfrak{g}$ (the closure in Zariski topology) consists of fin. many G-orbits.

Proof: Let $x \in \mathbb{O}$, and V be as above. Then $\overline{GL(V)x}$ consists of finitely many $GL(V)$-orbits (they all have the same char. polynomial). Then we argue as in the proof of Cor 1. $\square$

2) Algebra $\mathbb{C}[\widetilde{\mathbb{O}}]$ (reference: Sec 8 in [J]).

Let $\widetilde{\mathbb{O}}$ be a G-equivariant cover of an orbit $\mathbb{O} \subset \mathfrak{g}^*$. In particular, it's a smooth symplectic variety & $G \curvearrowright \widetilde{\mathbb{O}}$ is Hamiltonian w. moment map $\mu: \widetilde{\mathbb{O}} \twoheadrightarrow \mathbb{O} \hookrightarrow \mathfrak{g}^*$.

The goal of this section is to understand structures & properties of the Poisson algebra $\mathbb{C}[\widetilde{\mathbb{O}}]$. Here's the main result.

Theorem: $\mathbb{C}[\tilde{\mathbb{O}}]$ is finitely generated. If $\mathbb{O}$ is nilpotent, then $\exists$ algebra $\mathbb{Z}_{\geq 0}$-grading on $\mathbb{C}[\tilde{\mathbb{O}}]$ s.t. $\mathbb{C}[\tilde{\mathbb{O}}]_0 = \mathbb{C}$ & $\deg\{\cdot,\cdot\} = -2$.

2.1) Finite generation

Here we prove that $\mathbb{C}[\tilde{\mathbb{O}}]$ is finitely generated. The proof has two ingredients:

(I) $\overline{\mathbb{O}}$ contains finitely many G-orbits - Corollary 2 from Sec 1 & they are even dimensional (b/c they are symplectic).

(II) A variant of the "Zariski main theorem" for quasi-finite morphisms. Recall that a morphism of varieties $\varphi: Y \to X$ is called quasi-finite if $|\varphi^{-1}(x)| < \infty \ \forall x \in X$. Examples are provided by finite morphisms, open embeddings and compositions of such

Suppose now that φ is quasi-finite & dominant, Y is normal & X is affine. Then the statements we need are:

- the integral closure of $\mathbb{C}[X] \simeq \varphi^* \mathbb{C}[X]$ in $\mathbb{C}[Y]$, call it A, is a finitely generated $\mathbb{C}[X]$-module. It's also normal ($\mathbb{C}[Y]$ is integrally closed in its field of fractions, so A is integrally closed in $\mathrm{Frac}(A)$), so $\tilde{X} := \mathrm{Spec}\, A$ is a normal affine variety.

• Note that $\varphi: Y \to X$ factors as $Y \xrightarrow{\varphi_1} \tilde{X} \xrightarrow{\varphi_2} X$, where φ_2 is finite by the construction, and φ_1 turns out to be an <u>open embedding</u>.

We apply (II) to $Y = \tilde{\mathbb{O}}$, $X = \overline{\mathbb{O}}$ & $\varphi = \mu$ (viewed as a morphism to $\overline{\mathrm{im}\,\mu} = \overline{\mathbb{O}}$ - dominant & quasi-finite).

Lemma: We have $\mathbb{C}[\tilde{\mathbb{O}}] = \mathbb{C}[\tilde{X}]$. In particular, $\mathbb{C}[\tilde{\mathbb{O}}]$ is finitely generated.

Proof: Since $\tilde{X}$ is normal, it's enough to show that $\mathrm{codim}_{\tilde{X}}\, \tilde{X} \setminus \tilde{\mathbb{O}} \geq 2$, then we are done by the Hartogs thm. Since φ_2 is finite, it suffices to show $\varphi_2(\tilde{X} \setminus \tilde{\mathbb{O}}) \subset \overline{\mathbb{O}} \setminus \mathbb{O}$, & use that $\mathrm{codim}_{\overline{\mathbb{O}}}\, \overline{\mathbb{O}} \setminus \mathbb{O} \geq 2$ thx to (I). The containment is equivalent to $\tilde{\mathbb{O}} \xrightarrow{\sim} \tilde{X} \times_X \mathbb{O}$.

For this note that $\mu: \tilde{\mathbb{O}} \to \mathbb{O}$ is finite. This is a general fact: if $H \subset G$ are algebraic groups & $H' \subset H$ is a finite index subgroup, then $G/H' \to G/H$ is finite: one can assume $H' = H^\circ$, then $G/H^\circ \to G/H$ is the quotient morphism for the action of the finite group H/H° on G/H°, such morphisms are always finite.

To prove $\tilde{\mathbb{O}} \xrightarrow{\sim} \tilde{X} \times_X \mathbb{O}$ take $f \in \mathbb{C}[X]$ vanishing on $X \setminus \mathbb{O}$. Then the localization $\mathbb{C}[\tilde{X}]_{\varphi_2^* f}$ is the integral closure of $\mathbb{C}[X]_f$ in $\mathbb{C}[\tilde{\mathbb{O}}_{\varphi^*(f)}]$, $\tilde{\mathbb{O}}_{\varphi^*(f)} = \{x \in \tilde{\mathbb{O}} \mid [\varphi^*(f)](x) \neq 0\}$. But $\tilde{\mathbb{O}}_{\pi^*(f)}$ is finite over $\mathbb{O}_f$ (b/c $\tilde{\mathbb{O}}$ is finite over $\mathbb{O}$) & is normal (b/c it's smooth). So $\tilde{\mathbb{O}}_f \xrightarrow{\sim} \tilde{X}_f \Rightarrow \tilde{\mathbb{O}} \xrightarrow{\sim} \tilde{X} \times_X \mathbb{O}$. $\square$

Rem: By the construction, G acts on $\tilde{X}$ by automorphisms. It's an algebraic group action (i.e the action map $G \times \tilde{X} \to \tilde{X}$ is a morphism): for example, b/c $G \curvearrowright \mathbb{C}[\tilde{\mathbb{O}}]$ is rational. Note that $\tilde{\mathbb{O}}$ is a unique open orbit in $\tilde{X}$.

2.2) Grading.

Here we assume that $\mathbb{O}$ is nilpotent. Our goal is to establish a grading on $\mathbb{C}[\tilde{\mathbb{O}}]$ as in the Thm. Gradings are very closely related to $\mathbb{C}^\times$-actions, so we start with the latter. First, let's describe the group of equivariant automorphisms of a homogeneous space.

Exercise 1: Let $H \subset G$ be algebraic groups. Then the group of G-equivariant (variety) automorphisms $Aut_G(G/H)$ is identified w. $N_G(H)/H$ acting on G/H by $n.gH \mapsto gn^{-1}H$.

Pick $e \in \mathbb{O}$ and include it into an $\mathfrak{sl}_2$-triple hence getting a homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$ and hence $SL_2 \to G$. Consider the composition $\gamma: \mathbb{C}^\times \xrightarrow{t \mapsto diag(t,t^{-1})} SL_2 \longrightarrow G$.

Exercise 2: $\gamma(\mathbb{C}^\times)$ normalizes $Z_G(e)$ and any of its finite index subgroups H.

Now pick $\tilde{e} \in \tilde{\mathbb{O}}$ mapping to e and let $H := G_{\tilde{e}}$ be its stabilizer. We get the action of $\mathbb{C}^\times$ on G/H by $t.(gH) = g\gamma(t)^{-1}H$ (by Exer 1) and the similar action on $\mathbb{O}$: $t.(ge) = g\gamma(t)^{-1}e$ (where $\gamma(t)^{-1}$ is viewed as an element of G).

The $\mathbb{C}^\times$-action on $\tilde{\mathbb{O}} \rightsquigarrow$ a rational $\mathbb{C}^\times$-action on $\mathbb{C}[\tilde{\mathbb{O}}]$

$\longleftrightarrow$ an algebra grading $\mathbb{C}[\tilde{\mathbb{O}}] = \bigoplus_{i \geq 0} \mathbb{C}[\tilde{\mathbb{O}}]_i$ w.

$\mathbb{C}[\tilde{\mathbb{O}}]_i = \{f \in \mathbb{C}[\tilde{\mathbb{O}}] \mid t.f = t^i f\}$.

Lemma: $\mathbb{C}[\tilde{\mathbb{O}}]_i = \{0\}$ for $i<0$ & $= \mathbb{C}$ for $i=0$. Moreover, $\deg \{\cdot,\cdot\} = -2$.

Proof: The dominant morphisms $\tilde{\mathbb{O}} \xrightarrow{\mu} \mathbb{O} \hookrightarrow \overline{\mathbb{O}}$ give rise to algebra embeddings $\mathbb{C}[\overline{\mathbb{O}}] \subset \mathbb{C}[\mathbb{O}] \subset \mathbb{C}[\tilde{\mathbb{O}}]$. Since $\tilde{\mathbb{O}} \to \mathbb{O}$ is $\mathbb{C}^\times$-equivariant, the inclusion $\mathbb{C}[\mathbb{O}] \subset \mathbb{C}[\tilde{\mathbb{O}}]$ is $\mathbb{C}^\times$-equivariant, hence graded.

Let's describe the $\mathbb{C}^\times$-action on $\mathbb{O}$. We have $t.(ge) = g\gamma(t)^{-1}e$. The Lie algebra homomorphism $d_1\gamma: \mathbb{C} \to \mathfrak{g}$ is given by $1 \mapsto h$ & $[h,e]=2e$. Hence $\gamma(t)^{-1}e = t^{-2}e$ & $t.(ge) = t^{-2}ge$. In particular, $\mathbb{O} \hookrightarrow \mathfrak{g}^*$ is $\mathbb{C}^\times$-equivariant for the dilation action. Hence $\mathbb{C}^\times \curvearrowright \mathbb{O}$ extends to $\overline{\mathbb{O}}$, and $\mathbb{C}[\overline{\mathbb{O}}] \subset \mathbb{C}[\mathbb{O}]$ is a graded subalgebra. Note that $\mathbb{C}[\overline{\mathbb{O}}]$ is a graded quotient of $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g})$ w. $\deg \mathfrak{g} = 2$. In particular, $\mathbb{C}[\overline{\mathbb{O}}]_0 = \mathbb{C}$, $\mathbb{C}[\overline{\mathbb{O}}]_{<0} = \{0\}$.

We'll prove $\mathbb{C}[\tilde{\mathbb{O}}]_0 = \mathbb{C}$ and leave $\mathbb{C}[\tilde{\mathbb{O}}]_{<0} = \mathbb{C}$ as an exercise. Note that $\tilde{\mathbb{O}}$ is an orbit of an irreducible algebraic group, hence is irreducible $\Rightarrow \mathbb{C}[\tilde{\mathbb{O}}]$ is a domain. Also it's integral over $\mathbb{C}[\bar{\mathbb{O}}]$. For $f \in \mathbb{C}[\tilde{\mathbb{O}}]$ $\exists\, a_0, \dots a_{k-1} \in \mathbb{C}[\bar{\mathbb{O}}] \mid f^k + a_{k-1} f^{k-1} + \dots + a_0 = 0$. By passing to deg 0 component of this equation, can assume $\deg a_i = 0\ \forall i \Rightarrow a_i \in \mathbb{C}$. Since $\mathbb{C}[\tilde{\mathbb{O}}]_0$ is a domain, get $k=1$ so $f \in \mathbb{C}$.

Finally, we prove $\deg \{\cdot,\cdot\} = -2 \Leftrightarrow t.\{\cdot,\cdot\} = t^{-2}\{\cdot,\cdot\}$ $\forall t \in \mathbb{C}^\times$ (where $\{\cdot,\cdot\}$ is viewed as a map $\mathbb{C}[\tilde{\mathbb{O}}] \times \mathbb{C}[\tilde{\mathbb{O}}] \to \mathbb{C}[\tilde{\mathbb{O}}]$) Recall (Sec 1.1 of Lec 3) that the bivector on $\tilde{\mathbb{O}}$ is lifted from that on $\mathbb{O}$. So it's enough to show $t.\{\cdot,\cdot\} = t^{-2}\{\cdot,\cdot\}$ on $\mathbb{C}[\mathbb{O}]$. Since $\mathbb{O}$ is open in $\bar{\mathbb{O}}$, what we need to show is that $\deg\{f,g\} = \deg f + \deg g - 2$ for homogeneous $f, g \in \mathbb{C}[\bar{\mathbb{O}}]$. This algebra is generated by the image of $\mathfrak{g}$ (living in deg 2) so can assume $f = \xi$, $g = \eta \in \mathfrak{g}$. But the map $\mathfrak{g} \to \mathbb{C}[\mathbb{O}]$ (induced by $\mathbb{O} \hookrightarrow \mathfrak{g}^*$) is the comoment map, hence $\{\xi,\eta\} = [\xi,\eta]$ viewed as function on $\mathbb{O}$. The degree of $[\xi,\eta]$ is 2 and $\deg\{f,g\} = \deg f + \deg g - 2$ follows.

Rem: The argument shows that on $\mathbb{C}[\mathcal{O}]$ we have a grading w. $\mathbb{C}[\mathcal{O}]_0 = \mathbb{C}$, $\mathbb{C}[\mathcal{O}]_{<0} = \{0\}$ & $\deg \{\cdot,\cdot\} = -1$.