# Lecture 8.

1) Filtered Poisson deformations.
2) Singular symplectic varieties.
3) Spec $\mathbb{C}[\mathbb{O}]$ is singular symplectic.

Ref: [Be].

## 1) Filtered Poisson deformations.

In Sec 2.1 of Lec 3 we have introduced the notion of a filtered quantization of a $\mathbb{Z}_{\geqslant 0}$-graded Poisson algebra $A$. Now we will introduce its classical counterpart.

**Definition**: Let $A$ be a $\mathbb{Z}_{\geqslant 0}$-graded Poisson algebra w. $\deg\{\cdot,\cdot\}=-d$ (for $d\in\mathbb{Z}_{\geqslant 0}$). By its filtered Poisson deformation we mean a pair $(\mathcal{A}^\circ, \iota)$, where

- $\mathcal{A}^\circ$ is a Poisson algebra equipped with an algebra filtration $\mathcal{A}^\circ=\bigcup\limits_{i\geqslant 0}\mathcal{A}^\circ_{\leqslant i}$ s.t. $\{\mathcal{A}^\circ_{\leqslant i},\mathcal{A}^\circ_{\leqslant j}\}\subset \mathcal{A}^\circ_{\leqslant i+j-d}$ $\forall i,j$. Note that this gives rise to a deg $-d$ $\{\cdot,\cdot\}$ on $\operatorname{gr}\mathcal{A}^\circ$:

$$\{a+\mathcal{A}^\circ_{\leqslant i-1}, b+\mathcal{A}^\circ_{\leqslant j-1}\}:=\{a,b\}+\mathcal{A}^\circ_{\leqslant i+j-d-1}\ (a\in\mathcal{A}^\circ_{\leqslant i}, b\in \mathcal{A}^\circ_{\leqslant j-1})$$

- $\iota: \operatorname{gr}\mathcal{A}^\circ \xrightarrow{\sim} A$, iso of graded Poisson algebras.

Similarly to the notion of isomorphism of filtered quantizations (Sec 2.3 of Lec 4) we can talk about isomorphisms of filtered Poisson deformations.

Goal: Assume $A$ is positively graded (meaning that $A_0 = \mathbb{C}$ & $A_i = 0$ for $i<0$) & fin. generated. Classify filtered quantizations & filtered Poisson deformations of $A$ (up to iso). We are mostly interested in $A = \mathbb{C}[\tilde{\mathbb{O}}]$ for equiv. covers $\tilde{\mathbb{O}}$ of nilpotent orbits (see Thm in Sec 2 of Lec 7).

## 2) Singular symplectic varieties.

One should not expect this problem to have a reasonable answers unless one imposes additional restrictions on $A$. Here is one restriction that can be imposed.

Definition 1 ([Be]): Let $X$ be a normal & irreducible Poisson variety (i.e. $\mathcal{O}_X$ is equipped with a Poisson bracket). We say that $X$ has symplectic singularities (is singular symplectic or just symplectic) if conditions (1) & (2) below hold:

(1) The Poisson structure on $X^{reg}$ (the smooth locus) is non-degenerete. Let $\omega^{reg}$ denote the corresponding symplectic form.

(2) There's a resolution of singularities $Y \xrightarrow{\pi} X$ ($Y$ is smooth, $\pi$ is proper & birational) s.t. $\pi^*\omega^{reg} \in \Gamma(\pi^{-1}(X^{reg}), \Omega^2_Y)$ extends to $Y$, i.e. is the restriction of some $\tilde{\omega} \in \Gamma(Y, \Omega^2_Y)$ ($\tilde{\omega}$ is automatically unique).

Remarks: 1) As Beauville checked in Sec 1.2 of [Be], condition (2) is equivalent to the stronger condition: the conclusion of (2) holds for <u>all</u> resolutions.

2) $\tilde{\omega}$ is closed but we don't require $\tilde{\omega}$ to be non-degenerate ($\Leftrightarrow$ symplectic). If it is, then we say that $\pi: Y \to X$ is a symplectic resolution of singularities.

Definition 2: Let $A$ be fin. gen'd positively graded Poisson algebra. If $X := \mathrm{Spec}(A)$ is singular symplectic, then we say that $X$ is a conical symplectic singularity.

## 2) Spec $\mathbb{C}[\mathbb{O}]$ is singular symplectic

Let $G$ be a s/simple group. Let $\tilde{\mathbb{O}}$ be a $G$-equivariant cover of a nilpotent orbit in $\mathfrak{g}$. In Sec 2 of Lec 7 we've seen that $\mathbb{C}[\tilde{\mathbb{O}}]$ is fin. gen'd positively graded Poisson algebra. We'll see that $\mathrm{Spec}(\mathbb{C}[\tilde{\mathbb{O}}])$ is singular symplectic (and hence a conical symplectic singularity). Here we handle the case when $\tilde{\mathbb{O}}=\mathbb{O}\subset\mathfrak{g}^*$; the general case will be covered in the next lecture.

**Lemma 1:** $X=\mathrm{Spec}\,\mathbb{C}[\tilde{\mathbb{O}}]$ satisfies (1) $\forall$ cover $\tilde{\mathbb{O}}$ of an adjoint orbit.

Proof: In the proof of the lemma in Sec. 2.1 of Lec 7, we've seen that $\mathrm{codim}_X X\setminus\tilde{\mathbb{O}}\geqslant 2$. Hence $\mathrm{codim}_{X^{reg}} X^{reg}\setminus\tilde{\mathbb{O}}\geqslant 2$. Let $\mathcal{P}\in\Gamma(X^{reg},\Lambda^2 T_{X^{reg}})$ be the Poisson bivector. $\tilde{\mathbb{O}}$ is symplectic, so $\mathcal{P}_x$ is non-degenerate $\forall\, x\in\tilde{\mathbb{O}}$.

Let $n=\frac{1}{2}\dim\tilde{\mathbb{O}}$. For $x\in X^{reg}$, the bivector $\mathcal{P}_x$ is degenerate $\Leftrightarrow \Lambda^n\mathcal{P}_x=0$. But $\Lambda^n\mathcal{P}\in\Gamma(X^{reg},\overbrace{\Lambda^{2n}T_{X^{reg}}}^{\text{line bundle}})$. The zero locus of a section of a line bundle has pure codim 1. Thx to $\mathrm{codim}_{X^{reg}}X^{reg}\setminus\tilde{\mathbb{O}}\geqslant 2$, the zero locus of $\Lambda^n\mathcal{P}$ is empty. $\square$

In the rest of the section we'll check (2) for $X = \operatorname{Spec}\mathbb{C}[\mathbb{O}]$ by explicitly constructing a resolution of $X$. Let $(e,h,f)$ be $\mathfrak{sl}_2$-triple w. $e\in\mathbb{O}$.

Recall the decomposition $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, $\mathfrak{g}_i=\{x\in\mathfrak{g}\mid [h,x]=ix\}$. Consider $\mathfrak{g}_{\geq 2}\subset\mathfrak{g}_{\geq 0}$. Note that:

- $\mathfrak{g}_{\geq 0}=\mathfrak{g}_0\ltimes\mathfrak{g}_{\geq 1}$, where $\mathfrak{g}_{\geq 1}$ consists of nilpotent elements. We have $\mathfrak{g}_0=\operatorname{Lie}(G_0)$, where $G_0=Z_G(h)$, a Levi subgroup. Then $P:=G_0\ltimes\exp(\mathfrak{g}_{\geq 1})$, an algebraic subgroup of $\mathfrak{g}$, connected b/c so is $G_0$.
- $\mathfrak{g}_{\geq 2}$ is an ideal in $\mathfrak{g}_{\geq 0}$, hence preserved by $G_{\geq 0}$.

Set $Y:=G\times^P\mathfrak{g}_{\geq 2}$. By definition, this is the quotient of $G\times\mathfrak{g}_{\geq 2}$ by the action of $P$ given by $p.(g,x)=(gp^{-1},p.x)$ (see Sec 4.8 in [PV] for the construction). The $P$-orbit of $(g,x)$ – a point in $Y$ – will be denoted by $[g,x]$. The map $[g,x]\mapsto gP$ realizes $Y$ as a $G$-equivariant vector bundle over $G/P$. Furthermore the action map $G\times\mathfrak{g}_{\geq 2}\to\mathfrak{g}$, $(g,x)\mapsto gx$, is $P$-invariant, hence gives a well-defined map

$$\pi: Y\longrightarrow\mathfrak{g},\ [g,x]\mapsto gx, \tag{1}$$

Here is our main result.

Thm: (1) $\pi$ is a projective morphism.

(2) $\operatorname{im} \pi = \overline{\mathbb{O}}$.

(3) $\pi: Y \longrightarrow \overline{\mathbb{O}}$ is a resolution of singularities.

(4) it factors through $X = \operatorname{Spec} \mathbb{C}[\mathbb{O}]$.

(5, Panyushev) Let $\omega_{KK}$ be the Kirillov-Kostant form on $\mathbb{O}$. Then $\pi^*\omega_{KK}$ extends to $Y$ (this checks condition (2) of Def'n 1 in Sec 2).

Proof of Theorem

(1): The subalgebra $\mathfrak{g}_{\geq 0}$ is parabolic (meaning that it contains a Borel subalgebra), so $P \subset G$ is a parabolic subgroup, and hence $G/P$ is projective (a parabolic flag variety for $G$). See [OV], Exer. 20-27 for Sec 4.2.

Now note that we have factorization

$$\begin{tikzcd} G\times^P \mathfrak{g}_{\geq 2} \arrow[rr] \arrow[dr, "\text{induced by } \mathfrak{g}_{\geq 2}\hookrightarrow \mathfrak{g}"'] & & \mathfrak{g} \\ & G\times^P \mathfrak{g} \arrow[ur, "{[g,x]\mapsto gx}"'] & \end{tikzcd}$$

Further we have the following commutative diagram

$$\begin{tikzcd}
G\times\mathfrak{g} \arrow[rr, "\sim", "(g,x)\mapsto(g,\underset{\text{P-invariant!}}{gx})"'] \arrow[d, "(g,x)\mapsto{[g,x]}"'] & & G\times\mathfrak{g} \arrow[d, "(g,x)\mapsto(gP,x)"] \\
G\times^P\mathfrak{g} \arrow[rr, "\sim"] \arrow[dr, "{[g,x]}\mapsto gx"'] & & G/P\times\mathfrak{g} \arrow[dl, "(gP,x)\mapsto x"] \\
& \mathfrak{g} &
\end{tikzcd}$$

Since $G/P$ is projective, the $\swarrow$ arrow is projective. So is the $\searrow$ arrow. This proves (1).

(2): We claim that $Pe$ is an open orbit in $\mathfrak{g}_{\geq 2}$. It's enough to show $T_e Pe = \mathfrak{g}_{\geq 2}$. But $T_e Pe = [\mathfrak{p}, e] = [e, \mathfrak{g}_{\geq 0}] = [\text{rep. th. of } \mathfrak{sl}_2] = \mathfrak{g}_{\geq 2}$.

Since $\operatorname{im}\pi = G\mathfrak{g}_{\geq 2}$, we see that $\mathbb{O} = Ge$ is dense in $\operatorname{im}\pi$. Since $\pi$ is projective, $\operatorname{im}\pi$ is closed, giving $\operatorname{im}\pi = \overline{\mathbb{O}}$.

(3) We claim that $\pi: G\times^P Pe \xrightarrow{\sim} \mathbb{O}$. We have $G\times^P Pe \simeq G/Z_P(e)$ so it's enough to show $Z_P(e) = Z_G(e)$.
First, note $\mathfrak{z}_{\mathfrak{g}}(e) \subset \mathfrak{g}_{\geq 0} = \mathfrak{p} \Rightarrow \mathfrak{z}_{\mathfrak{p}}(e) = \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{p} = \mathfrak{z}_{\mathfrak{g}}(e)$.
Then, by Sec 1 of Lec 6, $Z_G(e) = Z_G(e,h,f) \ltimes Z_+$. The subgroup $Z_+$ is connected, so $\mathfrak{z}_{\mathfrak{p}}(e) = \mathfrak{z}_{\mathfrak{g}}(e) \Rightarrow Z_+ \subset Z_P(e)$. And $Z_G(e,h,f) \overset{=\text{ in fact}}{\subset} Z_G(e) \cap Z_G(h) = Z_G(e) \cap G_0 \subset Z_G(e) \cap P = Z_P(e)$.

So $Z_G(e) = Z_P(e)$, and $\pi$ is birational. Combining this with (1), we see that $\pi$ is a resolution of singularities.

(4): Note that $X$ is the normalization of $\overline{\mathbb{O}}$. Now we use the following: $\forall$ dominant morphism $Y \to X'$, where $Y$ is normal factors through the normalization of $X'$.

(5) Let $\alpha \in \mathfrak{p}$. We can identify $T_{(1,\alpha)}Y$ with $\mathfrak{g}_{<0} \oplus \mathfrak{g}_2$ via $(x,y) \mapsto x_{Y,(1,\alpha)} + y$, where $y$ is viewed as a tangent vector to the fiber. Here $x_Y$ is the image of $x \in \mathfrak{g}$ under the homomorphism $\mathfrak{g} \longrightarrow \mathrm{Vect}(Y)$ induced by the $G$-action.

Exercise 1: Check that the map $\mathfrak{g}_{<0} \oplus \mathfrak{g}_{\geq 2} \to T_{(1,\alpha)}Y$ is an iso

Hint: since $\mathfrak{g} = \mathfrak{g}_{<0} \oplus \mathfrak{p}$, the map $\mathfrak{g}_{<0} \to T_1(G/P)$, $x \mapsto x_{G/P,1}$ is an iso, then use SES $0 \to \mathfrak{g}_{\geq 2} \longrightarrow T_{(1,\alpha)}(G \times^P \mathfrak{g}_{\geq 2}) \longrightarrow T_1(G/P) \to 0$.

For $\alpha \in Pe$, we want to compute $d\pi^*_{(1,\alpha)}\omega_{KK,\alpha}$.

Claim: for $(x,y), (u,v) \in \mathfrak{g}_{<0} \oplus \mathfrak{g}_2$, we have:

Killing form

$$d\pi^*_{(1,\alpha)}\omega_{KK,\alpha}((x,y),(u,v)) = (\alpha, [x,u]) + (x,v) - (y,u) \qquad (*)$$

Let's explain why we need the claim. For $\alpha \in \mathfrak{g}_{\geq 2}$, define $\tilde{\omega}_{(1,\alpha)} \in \Lambda^2 T^*_{(1,\alpha)} Y$ as $(*)$ – this makes sense even if $\alpha \notin Pe$.

Exercise 2: $\exists!$ $G$-invariant 2-form $\tilde{\omega}$ on $Y$ whose value at $(1,\alpha)$ is $\tilde{\omega}_{(1,\alpha)}$. It extends $\pi^* \omega_{KK}$ proving (5).

Proof of Claim:

Recall that $\omega_{KK,\alpha}([\xi,\alpha],[\eta,\alpha]) = (\alpha,[\xi,\eta])$, $\forall \xi,\eta \in \mathfrak{g}$.
We have $d\pi_{(1,\alpha)}(x,y) = [x,\alpha]+y$. Note that $\exists$ $y' \in \mathfrak{g}_{\geq 0}$ w. $y=[y',\alpha]$. Indeed, $\alpha = pe$ for $p \in P$. The subspaces $\mathfrak{g}_{\geq 2} \subset \mathfrak{g}_{\geq 0}$ are $P$-stable. We have $p^{-1}y \in \mathfrak{g}_{\geq 2} \subset \operatorname{im}[e,\cdot]$, so $\exists$ $y'' \in \mathfrak{g}_{\geq 0}$ w. $p^{-1}y = [y'',e]$. Set $y' := py''$. Let $[\xi,\alpha] = [x,\alpha]+y$: we can find $x \in \mathfrak{g}_{<0}, y \in \mathfrak{g}_{\geq 2}$ e.g. b/c $\operatorname{im} d\pi_{(1,\alpha)} = T_\alpha \mathbb{O} = [\mathfrak{g},\alpha]$, so that we can take $\xi = x+y'$. Let $[\eta,\alpha] = [u,\alpha]+v$, $u \in \mathfrak{g}_{<0}, v \in \mathfrak{g}_{\geq 2}$.
Then $d\pi^*_{(1,\alpha)} \omega_{KK,\alpha}((x,y),(u,v)) = (\alpha,[\xi,\eta]) = (\xi,[\eta,\alpha]) =$
$= (x+y',[u,\alpha]+v) = \left((y',v)=0 \text{ b/c } y' \in \mathfrak{g}_{\geq 0}, v \in \mathfrak{g}_{\geq 2}\right) =$
$= (x,[u,\alpha]) + (x,v) + (y',[u,\alpha]) = [(y',[u,\alpha]) = -([y',\alpha],u)$
$= -(y,u)] = (\alpha,[x,u]) + (x,v) - (y,u)$. $\square$