Lecture 9.

1) Singularity Zoo
2) Basics of Invariant theory

[Be]; [E], Secs 17-19, 21; [L2]; [PV], Secs 3.1-3.4, 4.1-4.4; [S]

1) Singularity Zoo

For a nilpotent orbit $\mathbb{O}$, we have proved that $\operatorname{Spec} \mathbb{C}[\mathbb{O}]$ is singular symplectic using a Lie theoretic construction. Our next goal is to generalize this to $\operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$ for an equivariant cover $\tilde{\mathbb{O}}$ of a nilpotent orbit. The proof here is an Algebro-geometric lemma based on relations between different kinds of singularities. In this section, we'll give a brief discussion of Cohen-Macaulay, Gorenstein & rational singularities. Then we'll prove the lemma.

1.1) Cohen-Macaulay schemes (Ref: [E], Secs 17-19)

Let R be a commutative (associative, unital) ring, and

M be a finitely generated R-module

Definition: An *M-regular sequence* in R is a collection $f_1, \dots, f_k \in R \mid f_i$ is not a zero divisor in $M/(f_1, \dots, f_{i-1})M$ $\forall i$; and $(f_1, \dots, f_k)M \neq M$.

If $M = R$, we just talk about regular sequences.

In fact, this is independent of the order of $f_1, \dots, f_k$.

Now let X be a finite type scheme$/\mathbb{C}$, and $\mathcal{M} \in Coh(X)$. Pick $x \in X$ & write $\mathcal{O}_{X,x}$, $\mathcal{M}_x$ for the stalks of $\mathcal{O}_X$ & $\mathcal{M}$ at x.

Definition: • The *depth*, $d(\mathcal{M}_x)$, of $\mathcal{M}_x$ is the maximal number of elements in an $\mathcal{M}_x$-regular sequence in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

• $\mathcal{M}$ is *maximal Cohen-Macaulay* (MCM) at x if $d(\mathcal{M}_x) = \dim \mathcal{O}_{X,x}$ $(= \dim_x X)$; $\mathcal{M}$ is *MCM* if it's MCM at $\forall x \in X$.

• X is *Cohen-Macaulay* -CM- (at x) if $\mathcal{O}_X$ is MCM (at x).

Remark: By [E], Theorem 17.4, every <u>maximal</u> $\mathcal{M}_x$-regular sequence in $\mathfrak{m}_x$ has the same number of elements.

Examples: 1) Every smooth variety is CM: for $x \in X$, take $f_1, \dots, f_n \in \mathcal{O}_{X,x}$ s.t. $d_x f_1, \dots, d_x f_n \in T^*_{X,x}$ is a basis. This is a regular sequence. Further, $\mathcal{M}$ is MCM iff it's a vector bundle (see [E], Thm 19.9)

2) Let $\tilde{X}$ be affine & smooth. Let $f_1, \dots, f_k \in \mathbb{C}[X]$ be a regular sequence, and $X \subset \tilde{X}$ be a subscheme given by $f_1, \dots, f_k$ (the condition of being regular means $\operatorname{codim}_{\tilde{X}} X = k$). Then X is CM. We say that X is a complete intersection.

1.2) Gorenstein schemes ([E], Sec 21)

In the study of smooth varieties an important role is played by a canonical bundle (= the bundle of top forms). It turns out this can be generalized to any (finite type, at least) schemes - to the dualizing complex in the derived category. The Cohen-Macaulay schemes are characterized by the property that this complex sits in a single homological degree (= dim X). Gorenstein schemes we are going to introduce are characterized by the property that it's a line bundle.

For the sake of completeness, let's give a self-contained definition (that we are not going to use). Let X be a finite type CM scheme, $x\in X$, $R:=\mathcal{O}_{X,x}$.

Definition: An MCM R-module W is called *canonical* if $R \xrightarrow{\sim} End_R(W)$ & W admits a finite <u>injective</u> resolution.

By [E], Sec. 21.6, W exists and is unique.

Definition: X is *Gorenstein* if $\forall x$, the canonical $\mathcal{O}_{X,x}$-module is $\simeq \mathcal{O}_{X,x}$.

Example ([E], Sec. 21.8): Smooth schemes & local complete intersections are Gorenstein.

1.3) Rational singularities.

Here is a strengthening of the CM property that will play an important role in what follows. Let X be a normal finite type scheme/$\mathbb{C}$.

Definition: X has rational singularities if $\exists (\Leftrightarrow \forall)$ resolution of singularities $\varphi: Y \to X$ s.t. $R^i\varphi_* \mathcal{O}_Y = 0$, $\forall\, i>0$.

Remarks: 1) $\varphi_* \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X$ b/c X is normal ([H], Ch. 3, Sec 11)

2) smooth $\Rightarrow$ rational singularities $\overset{\text{Kempf}}{\Rightarrow}$ CM.

1.4) Relations to symplectic singularities.

Theorem 1 ([Be], Proposition 1.3): Any singular symplectic variety is Gorenstein & has rational singularities.

The following theorem is a partial converse.

Theorem 2 (Namikawa): Suppose that X is Gorenstein & has rational singularities. Also suppose that X^{reg} has a symplectic form. Then X is singular symplectic.

Rem: Let's sketch why symplectic $\Rightarrow$ rational. Let X be an normal CM variety. Kempf proved the following:

X has rational singularities $\Leftrightarrow \exists (\Leftrightarrow \forall)$ resolution of singularities $Y \xrightarrow{\pi} X$ s.t. $\forall$ top form σ on X^{reg}, $\pi^*(\sigma)$ extends to Y. Now if X^{reg} is symplectic, we have that $\Omega^{2n}_{X^{reg}}$ is a free rk 1 $\mathcal{O}_{X^{reg}}$-module w. basis $\Lambda^n \omega^{reg}$ ($n = \frac{1}{2}\dim X$). If X is symplectic, then $\pi^*\omega^{reg}$ extends to Y and, hence, so does any top form. So X has rational singularities. Therefore, having symplectic singularities is the natural strengthening of having rational singularities in the setting of Poisson varieties.

1.5) Spec $\mathbb{C}[\tilde{\mathbb{O}}]$ is singular symplectic.

The argument below is [L2], Lemma 2.5. Recall that we assume that $\tilde{\mathbb{O}}$ is a G-equivariant cover of a nilpotent orbit. Let $X = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$. By Lemma 1 in Section 2 in Lec 8, X^{reg} is symplectic. It turns out that X is Gorenstein w. rational singularities b/c $\operatorname{Spec} \mathbb{C}[\mathbb{O}]$ is, a result of Broer. We apply Theorem 2 from Sec 1.4 to finish the proof.

2) Basics of Invariant theory

The goal of this section is an express intro to Invariant theory.

2.1) Categorical quotients.

Let G be a reductive algebraic group (perhaps disconnected). Suppose that it acts (algebraically) on an affine variety X. The following result goes back to Hilbert, it uses that the rational G-reps are completely reducible.

Theorem 1: The algebra of invariants $\mathbb{C}[X]^G$ is finitely generated.

So we can form the variety $X/\!\!/G$ called the categorical quotient of X (by the action of G). The inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ gives rise to a dominant morphism $X \longrightarrow X/\!\!/G$ called the quotient morphism and denoted by π_G.

By the very definition, the pair $(X/\!\!/G, \pi_G)$ enjoys the following universal property that explains the name "categorical quotient"

Exercise: Let Y be an affine variety & $\varphi: X \to Y$ a G-invariant morphism. Then $\exists!\ \underline{\varphi}: X/\!/G \to Y$ w. $\varphi = \underline{\varphi} \circ \pi_G$.

The following important theorem describes basic properties of π_G. The proof is also based on the complete reducibility.

Theorem 2: The following are true:

1) If $Z \subset X$ is a G-stable closed subvariety, then $\pi_G(Z) \subset X/\!/G$ is closed, it's identified w. $Z/\!/G$.

2) If Z_1, Z_2 are closed & G-stable & $Z_1 \cap Z_2 = \emptyset$, then $\pi(Z_1) \cap \pi(Z_2) = \emptyset$.

3) $\forall\ y \in X/\!/G\ \exists!$ closed G-orbit in $\pi_G^{-1}(y)$.

So the points of $X/\!/G$ parameterize the <u>closed</u> G-orbits in X.

To finish this section we discuss the situation when G is finite. Here we write X/G instead of $X/\!/G$.

The proof of the next proposition is an exercise.

Proposition 1: If G is finite, then $\pi_G : X \longrightarrow X/G$ is finite & every fiber is a single G-orbit.

We can (to some extent) describe the "local structure" of $X//G$. We'll do this when G is finite.

Proposition 2: Let $H \subset G$ be a subgroup. The locus of $z \in X/H$ s.t. the natural morphism $\pi : X/H \longrightarrow X/G$ is etale at z consists exactly of the H-orbits of $x \in X$ s.t. $G_x \subset H$.

This proposition tells us that etale locally near $y \in X/G$ the quotient X/G looks like X/G_x near $G_x x$ for $x \in X$ w. $Gx = y$.

2.2) Properties of quotients.

One can start by asking for which G-actions on a smooth affine X, the quotient $X//G$ is smooth. Let's address this question in the case when $X = V$ is a vector space w. linear G-action (the general case reduces to here) & G is finite (this is an essential restriction).

Definition: A complex reflection in $GL(V)$ is a finite order element $s \in GL(V)$ s.t. $\mathrm{rk}(s - \mathrm{id}) = 1$. A finite subgroup in $GL(V)$ is called a complex reflection group if it's generated by complex reflections.

Example: Every (real) reflection group is a complex reflection group. This applies, for example, to a Weyl group W acting on a Cartan subalgebra $\mathfrak{h}$.

Theorem (Chevalley-Shephard-Todd) Let $G \subset GL(V)$ be a finite group. TFAE:

(a) G is a complex reflection group.

(b) V/G is smooth.

(c) $\mathbb{C}[V]$ is a free $\mathbb{C}[V]^G$-module.

The proof can be found in [B], Ch 5, Sec 5.

Rem: In fact, for any reductive group G, and any affine variety X w. rational singularities, then $X /\!/ G$ has rational singularities as well – a theorem of Boutot (see [PV], Sec 3.9)

If G is a finite subgroup in $GL(V)$ that doesn't contain complex reflections then V is Gorenstein $\Leftrightarrow$ $G \subset SL(V)$, [S], Sec 8.