LECTURE 4: SOERGEL'S THEOREM AND SOERGEL BIMODULES

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Abstract. These are notes for a talk given at the MIT-Northeastern Graduate Student Seminar on category $\mathcal{O}$ and Soergel bimodules, Fall 2017.

Contents

1. Goals and structure of the talk

The main goal of this talk is to introduce Soergel's $\mathbb{V}$-functor and study its properties. The exposition will be as follows. In Section 2 we define Soergel's $\mathbb{V}$-functor and state three theorems of Soergel. In Section 3 we will prove the first of them. For this purpose we will construct extended translation functors that naturally extend translation functor to bigger categories.

2. Soergel $\mathbb{V}$-functor

Let $\mathfrak{g}$ be a semisimple Lie algebra, W its Weyl group and $w_0 \in W$ the longest element. By $P_{min} := P(w_0 \cdot 0)$ we denote the projective cover of $L_{min} := L(w_0 \cdot 0)$.

Definition 2.1. *The Soergel $\mathbb{V}$-functor is a functor between the principal block $\mathcal{O}_0$ and the category of right modules over* $\operatorname{End}(P_{min})$ *given by* $\mathbb{V}(\bullet) = \operatorname{Hom}(P_{min}, \bullet)$.

We set $C := \mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^W_+)$ where $\mathbb{C}[\mathfrak{h}]^W_+ \subset \mathbb{C}[\mathfrak{h}]^W$ is the ideal of all elements without constant term and $(\mathbb{C}[\mathfrak{h}]^W_+) = \mathbb{C}[\mathfrak{h}]\mathbb{C}[\mathfrak{h}]^W_+$. This is called the coinvariant algebra. The main goal of the talk is to prove some properties of $\mathbb{V}$.

Theorem 2.2. $\operatorname{End}_{\mathcal{O}}(P_{min}) \simeq C$.

Theorem 2.3. $\mathbb{V}$ *is fully faithful on projectives.*

Theorem 2.4. $\mathbb{V}(\mathcal{P}_i\bullet) \simeq \mathbb{V}(\bullet) \underset{\mathbb{C}[\mathfrak{h}]^{s_i}}{\otimes} \mathbb{C}[\mathfrak{h}]$.

3. Endomorphisms of P_{min}

In this section we prove that $\mathrm{End}_{\mathcal{O}}(P_{min}) = C$.

Before we proceed to the proof we need to observe some properties of $\mathcal{O}_0$, P_{min} and C. This is done in next three subsections.

3.1. $\mathcal{O}_0$ is a highest weight category.

Recall that we have a Bruhat order on the Weyl group W. For an element $w \in W$ we say that $\underline{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$ is an expression of w if $w = s_{i_1} s_{i_2} \dots s_{i_k}$. The minimal number $l(w)$ of elements in the expression of w is called length of w. We say that the expression $\underline{w}$ is reduced if $l(w) = k$.

Definition 3.1. *Consider w_1, $w_2 \in W$. We say $w_1 \preceq w_2$ if there are reduced expressions $\underline{w_1}$ of w_1 and $\underline{w_2}$ of w_2 such that $\underline{w_1}$ is a subexpression of $\underline{w_2}$.*

In this subsection we will prove that the principal block $\mathcal{O}_0$ is a highest weight category. Let us recall the definition of a highest weight category from Daniil's talk.

Definition 3.2. *Consider an abelian category $\mathcal{C}$ which has a finite number of simple objects, enough projectives and every object has finite length (equivalently $\mathcal{C} \simeq A$-mod, where A is a finite dimensional associative algebra). The highest weight structure on such a category, is a partial order $\preceq$ on the set of simple objects $\mathrm{Irr}(\mathcal{C})$ and the set of standard objects Δ_L, $L \in \mathrm{Irr}(\mathcal{C})$ such that:*
- $\mathrm{Hom}_C(\Delta_L, \Delta_{L'}) \neq 0 \Rightarrow L \preceq L'$ *and* $\mathrm{End}_C(\Delta_L) = \mathbb{C}$.
- *The projective cover P_L of L admits an epimorphism onto Δ_L and $\mathrm{Ker}(P_L \to \Delta_L)$ admits a filtration by $\Delta_{L'}$ with $L \prec L'$.*

Proposition 3.3. *The category $\mathcal{O}_0$ is a highest weight category with respect to the opposite Bruhat order.*

Proof. Chris has proved that $P(w \cdot 0)$ is a direct summand in $\mathcal{P}_k \dots \mathcal{P}_1 \Delta(0)$. Note that all standards occurring in the bigger projective have labels $w' \preceq w$ in the Bruhat order and w appears only once. So $K := \mathrm{Ker}(P(w \cdot 0) \to \Delta(w \cdot 0))$ is filtered with $\Delta(w' \cdot 0)$ for $w' \prec w$.
It remains to show that $\mathrm{Hom}(\Delta(w \cdot 0), \Delta(w' \cdot 0) \neq 0 \Rightarrow w' \preceq w$. Note that in the opposite direction it was proved in the first talk of the seminar. If $\mathrm{Hom}(\Delta(w \cdot 0), \Delta(w' \cdot 0) \neq 0$ then the induced map on $L(w \cdot 0)$ is non-trivial, so $[\Delta(w' \cdot 0) : L(w \cdot 0)] \neq 0$. By BGG reciprocity $[\Delta(w' \cdot 0) : L(w \cdot 0)] = (P(w \cdot 0) : \Delta(w' \cdot 0))$, so $w' \preceq w$. □

3.2. Properties of P_{min}.

For the longest element $w_0 \in W$ we have the corresponding minimal element $\lambda_{min} := w_0 \cdot \lambda$. For that element we have $\Delta_{min} := \Delta(\lambda_{min}) \simeq L_{min} := L(\lambda_{min}) \simeq \nabla_{min} := \nabla(\lambda_{min})$. Let P_{min} be a projective cover of Δ_{min}. In his talk Chris defined translation functors $T_{\lambda \to \mu}$. In this talk we will be especially interested in translations to the most singular case when $\mu = -\rho$. Let us set a notation $\mathcal{O}_\lambda$ for $\mathcal{O}_{\chi_\lambda}$. Note that every object in $\mathcal{O}_{-\rho}$ is a direct sum of some copies of $\Delta(-\rho) = L(-\rho)$, so $\mathcal{O}_{-\rho}$ is equivalent to the category of vector spaces. We set $T := T_{\lambda \to -\rho}$ and $T^* := T_{-\rho \to \lambda}$. These functors are exact and biadjoint. We want to find a description of the projective cover P_{min} using translations functors.

Proposition 3.4. *$P_{min} = T^*(\Delta(-\rho))$.*

Proof. $\Delta(-\rho)$ is projective object in $\mathcal{O}_{-\rho}$ and the functor T^* is left adjoint to the exact functor T. Therefore $T^*(\Delta(-\rho))$ is projective. It is enough to show that $\dim \mathrm{Hom}(T^*(\Delta(-\rho)), L) = 1$ if $L = L_{min}$ and 0 else.

Let us compute $\mathrm{Hom}(T^*(\Delta(-\rho)), L)$. Since T^* is left adjoint to T we have $\mathrm{Hom}(T^*(\Delta(-\rho)), L) \simeq \mathrm{Hom}(\Delta(-\rho), T(L))$. From Chris's talk we know that $T(L_{min}) \simeq \Delta(-\rho)$ and $T(L) = 0$ for any other simple L that finishes the proof. □

Remark 3.5. *$\Delta(-\rho)$ is self-dual object in the category $\mathcal{O}_{-\rho}$ and T^* commutes with duality. Analogous to Proposition 3.4 statement shows that $P_{min} = T^*(\Delta(-\rho))$ is the injective envelope of Δ_{min}. In fact, there are no other projective-injective elements in $\mathcal{O}_\lambda$.*

Corollary 3.6. *$\mathbb{V}(\Delta(w\cdot 0))$ is a one-dimensional $\operatorname{End}(P_{min})$-module for any $w \in W$.*

We will show later that such module is unique.

Proof. $\mathbb{V}(\Delta(w\cdot 0)) = \operatorname{Hom}_{\mathcal{O}_0}(T^*\Delta(-\rho), \Delta(w\cdot 0)) = \operatorname{Hom}_{\mathcal{O}_{-\rho}}(\Delta(-\rho), T\Delta(w\cdot 0))$. Chris has proved in his talk that $T\Delta(w\cdot 0) = \Delta(-\rho)$. Therefore $\dim \mathbb{V}(\Delta(w\cdot 0)) = \dim \operatorname{Hom}_{\mathcal{O}_{-\rho}}(\Delta(-\rho), \Delta(-\rho)) = 1$. ☐

Recall from Daniil's talk the definition of a standard filtration.

Definition 3.7. *An object $M \in \mathcal{O}$ is standardly filtered if there is a chain of submodules $0 = F_0M \subset F_1M \subset F_2M \subset \ldots \subset F_nM = M$ such that each ${F_{i+1}M}/{F_iM}$ is isomorphic to a Verma module.*

Proposition 3.8. *Every Verma module $\Delta(w\cdot 0)$ appears in a standard filtration of P_{min} exactly one time.*

Proof. By BGG reciprocity we have
$(P_{min} : \Delta(w\cdot 0)) = [\Delta(w\cdot 0) : L_{min}] = [\Delta(w\cdot 0) : \Delta_{min}] = 1$ where the last equality was proved in the proof of Proposition 5 from the first lecture. ☐

3.3. **Properties of C.**

Lemma 3.9. *The following are true.*

1. *C is a local commutative algebra, in particular, it has a unique irreducible representation (we will just write $\mathbb{C}$ for that irreducible representation).*
2. *$C \cong H^*(G/B, \mathbb{C})$.*
3. *There is a nonzero element $\omega \in C$ such that for any other element $f_1 \in C$, there is $f_2 \in C$ with $f_1f_2 = \omega$.*
4. *The socle (=the maximal semisimple submodule) of the regular C-module C is simple, equivalently, by (a), $\dim \operatorname{Hom}(\mathbb{C}, C) = 1$.*
5. *We have an isomorphism $C \cong C^*$ of C-modules. In particular, C is an injective C-module.*

Proof. (1) is clear. To prove (2), let us recall that $H^*(G/B,\mathbb{C})$ is generated by $H^2(G/B,\mathbb{C}) \cong \mathfrak{h}^*$ (in particular, there's no odd cohomology and the algebra $H^*(G/B,\mathbb{C})$ is honestly commutative). So we have an epimorphism $\mathbb{C}[\mathfrak{h}] \twoheadrightarrow H^*(G/B,\mathbb{C})$. The classical fact is that the kernel is generated by $\mathbb{C}[\mathfrak{h}]^W_+$ so $C \xrightarrow{\sim} H^*(G/B,\mathbb{C})$.

Let us prove (3). We claim that this holds for the cohomology of any compact orientable manifold M. Indeed, $\dim H^{top}(M,\mathbb{C}) = 1$, let us write ω for the generator. The pairing $(\alpha,\beta) := \int_M \alpha\wedge\beta$ is nondegenerate on $H^*(M,\mathbb{C})$. (3) follows.

By (3) any nonzero C-submodule of C contains ω. This implies (4).

Let us prove (5). Consider the linear isomorphism $C \xrightarrow{\sim} C^*$ given by $(\cdot,\cdot)$. Note that the form $(\cdot,\cdot)$ is invariant: $(\gamma\alpha,\beta) = (\alpha,\gamma\beta)$. So the map $C \to C^*$ is C-linear. ☐

3.4. **Strategy of proof.** Our strategy of proving $\operatorname{End}_{\mathcal{O}}(P_{min}) = C$ is as follows.

1) We define a functor (an extended translation functor) $\tilde{T}_{0\to-\rho} : \mathcal{O}_0 \to C\text{-mod}$ with the property that $\mathsf{frg}\circ\tilde{T}_{0\to-\rho} = T_{0\to-\rho}$, where $T_{0\to-\rho} : \mathcal{O}_0 \to \mathcal{O}_{-\rho}$ is the usual translation functor, and $\mathsf{frg} : C\text{-mod} \to \mathrm{Vect}$ is the forgetful functor (recall from the beginning of Subsection 3.2 that $\mathcal{O}_{-\rho}$ is the semisimple category with a single simple object, so it is the category Vect of vector spaces).

2) Since $\mathcal{F} := \tilde{T}_{0\to-\rho}$ is an exact functor between categories that are equivalent to the categories of modules over finite dimensional algebras, it admits left and right adjoint functors to be denoted

by $\mathcal{F}^!, \mathcal{F}^*$. We will show that $P_{min} = \mathcal{F}^!(C) = \mathcal{F}^*(C)$. Therefore we have a natural map $C = \operatorname{End}_{C\text{-mod}}(C, C) \to \operatorname{End}_{\mathcal{O}}(\mathcal{F}^*(C), \mathcal{F}^*(C)) = \operatorname{End}_{\mathcal{O}}(P_{min})$.

3) We will establish an isomorphism $C \xrightarrow{\sim} \operatorname{End}(P_{min})$ by showing that $\mathcal{F}(P_{min}) = C$. A key ingredient for the latter is to show that $\mathcal{F}^*(\mathbb{C}) = \Delta(0)$.

3.5. **Extended translation functors.** The functor $\mathcal{F} = \tilde{T}_{0\to-\rho}$ is a special case of more general functors known as the extended translation functors.

Consider the dotted action of W on $\mathfrak{h}$. We set $\tilde{U} := U(\mathfrak{g}) \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]$ with the action of $\mathbb{C}[\mathfrak{h}]^W$ on $U(\mathfrak{g})$ by the Harish-Chandra isomorphism. Let J_λ be the maximal ideal corresponding to λ in $\mathbb{C}[\mathfrak{h}]$ and $I_{|\lambda|}$ the maximal ideal corresponding to $W \cdot \lambda$ in $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[\mathfrak{h}/W]$. We define $\tilde{U}\text{-mod}_\lambda$ as the category of finitely generated $\tilde{U}$-modules M such that $J_\lambda^n M = 0$ for n big enough. Let $\tilde{\mathcal{O}}_\lambda$ be a subcategory of $\tilde{U}\text{-mod}_\lambda$ consisting of $\tilde{U}$-modules M such that $M \in \mathcal{O}_\lambda$ considered as a $U(\mathfrak{g})$-module. The goal of this subsection is to construct and study an extended translation functor $\tilde{T}_{\lambda\to\mu} : \tilde{\mathcal{O}}_\lambda \to \tilde{\mathcal{O}}_\mu$.

Let $W_\lambda \subset W$ be a stabilizer of λ. We consider the algebra of W_λ-invariants $\tilde{U}^{W_\lambda} := U(\mathfrak{g}) \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]^{W_\lambda}$ and $J_\lambda^{W_\lambda} := J_\lambda \cap \mathbb{C}[\mathfrak{h}]^{W_\lambda}$. Let $\tilde{U}^{W_\lambda}\text{-mod}_\lambda$ be a category of finitely generated $\tilde{U}^{W_\lambda}$-modules M such that $(J_\lambda^{W_\lambda})^n M = 0$ for n big enough. We set $\tilde{\mathcal{O}}_\lambda^{W_\lambda}$ be a subcategory of $\tilde{U}^{W_\lambda}\text{-mod}_\lambda$ consisting of $\tilde{U}^{W_\lambda}$-modules M such that $M \in \mathcal{O}_\lambda$ considered as a $U(\mathfrak{g})$-module. We have natural restriction functors $\operatorname{Res}_\lambda : \tilde{\mathcal{O}}_\lambda \to \mathcal{O}_\lambda$ and $\operatorname{Res}_\lambda^{W_\lambda} : \tilde{\mathcal{O}}_\lambda^{W_\lambda} \to \mathcal{O}_\lambda$.

Proposition 3.10. *The functor* $\operatorname{Res}_\lambda^{W_\lambda}$ *is an equivalence of categories.*

Proof. The natural map $\mathfrak{h} \to \mathfrak{h}/W$ factors through $\mathfrak{h} \to \mathfrak{h}/W_\lambda \to \mathfrak{h}/W$. The map $\mathfrak{h}/W_\lambda \to \mathfrak{h}/W$ is unramified, so the formal neigbourhood of a point $W \cdot \lambda \in \mathfrak{h}/W$ is canonically isomorphic to a formal neighborhood of a point $W_\lambda \cdot \lambda \in \mathfrak{h}/W_\lambda$. In other words, $\varprojlim \mathbb{C}[\mathfrak{h}]^W / I_{|\lambda|}^n \simeq \varprojlim \mathbb{C}[\mathfrak{h}]^{W_\lambda} / (J_\lambda^{W_\lambda})^n$. Hence on any $M \in \mathcal{O}_\lambda$ we have an action of $\varprojlim \mathbb{C}[\mathfrak{h}]^{W_\lambda}/(J_\lambda^{W_\lambda})^n$ that makes M an object of $\tilde{\mathcal{O}}_\lambda^{W_\lambda}$. That gives a functor quasi-inverse to $\operatorname{Res}_\lambda^{W_\lambda}$. ☐

Remark 3.11. *The functor* $\operatorname{Res}_\lambda(\bullet)$ *has a natural left adjoint* $\operatorname{Ind}_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}}^{\mathbb{C}[\mathfrak{h}]}(\bullet)$ *and a natural right adjoint* $\operatorname{Hom}_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}}(\mathbb{C}[\mathfrak{h}], \bullet)$.

Corollary 3.12. *For* $\lambda + \rho$ *regular the functor* $\operatorname{Res}_\lambda$ *gives an equivalence of categories* $\tilde{\mathcal{O}}_\lambda$ *and* $\mathcal{O}_\lambda$.

Lemma 3.13. *For the most singular case we have* $\tilde{\mathcal{O}}_{-\rho} \simeq C\text{-mod}$.

Proof. Every object $M \in \mathcal{O}_{-\rho}$ is of form $M \simeq \Delta(-\rho) \otimes V$. Therefore the action of the central subalgebra $Z(U(\mathfrak{g})) = \mathbb{C}[\mathfrak{h}]^W$ on M factors through $\mathbb{C}[\mathfrak{h}]^W \to \mathbb{C}[\mathfrak{h}]^W / \mathbb{C}[\mathfrak{h}]^W_+ = \mathbb{C}$ and $\tilde{\mathcal{O}}_{-\rho}$ consists of $U_{-\rho} \otimes C$-modules from the category $\mathcal{O}_{-\rho}$. Therefore we have the functor $C\text{-mod} \to \tilde{\mathcal{O}}_{-\rho}$ given by $\Delta(-\rho) \otimes \bullet$ and the functor $\tilde{\mathcal{O}}_{-\rho} \to C\text{-mod}$ given by $\operatorname{Hom}_{U_{-\rho}}(\Delta(-\rho), \bullet)$. It is easy to check that these two functors are quasi-inverse. ☐

Therefore we have a translation functor $T_{\lambda\to\mu} : \tilde{\mathcal{O}}_\lambda^{W_\lambda} \to \tilde{\mathcal{O}}_\mu^{W_\mu}$. For integral λ, μ such that $\lambda + \rho$ and $\mu + \rho$ are dominant and $W_\lambda \subset W_\mu$ we want to extend it to the translation functor $\tilde{T}_{\lambda\to\mu} : \tilde{\mathcal{O}}_\lambda \to \tilde{\mathcal{O}}_\mu$. We claim that for $M \in \tilde{\mathcal{O}}_\lambda$ we have a natural structure of a $\tilde{U}$-module from the category $\tilde{\mathcal{O}}_\mu$ on $N := T_{\lambda\to\mu} \operatorname{Res}(M)$ constructed in the following way. We already have an action of $\tilde{U}_\mu^{W_\mu}$. Let $\rho_{\mu-\lambda}$ be an endomorphism of $\mathbb{C}[\mathfrak{h}]$ induced by the map $\rho_{\mu-\lambda}(x) = x + \mu - \lambda$ for $x \in \mathfrak{h}$. For $M \in \tilde{\mathcal{O}}_\lambda$ we have a natural action of $\mathbb{C}[\mathfrak{h}]$ by $\tilde{U}$-module endomorphisms on $\operatorname{Res}(M)$ that factors through $\mathbb{C}[\mathfrak{h}]/J_\lambda^n$ for n large enough. By the functoriality we have an action of $\mathbb{C}[\mathfrak{h}]$ on $N = T_{\lambda\to\mu} \operatorname{Res}(M)$. We twist this action with $\rho_{\mu-\lambda}$, so it factors through $\mathbb{C}[\mathfrak{h}]/J_\mu^n$. Let us denote this action as $z * m$.

Example 3.14. *Let $\mathfrak{g} = \mathfrak{sl}_2$. Let x be the generator of $\mathbb{C}[\mathfrak{h}]$. We are interested in the action of $\mathbb{C}[\mathfrak{h}]$ on $P(-2)$ and induced action on $\tilde{T}_{0\to -1}(P(-2))$. Let us choose a basis of $P(-2)$, for which the weight diagram is as below.*

$$\begin{tikzcd}
v_0 \arrow[d, bend right] & \\
v_{-2} \arrow[d, bend right] & w_{-2} \arrow[ul] \arrow[d, bend left] \\
v_{-4} \arrow[u] \arrow[d, bend right] & w_{-4} \arrow[ul] \arrow[u] \arrow[d, bend left] \\
\cdots \arrow[u] & \cdots \arrow[ul] \arrow[u]
\end{tikzcd}$$

The Casimir element $x^2 + 2x \in \mathbb{C}[\mathfrak{h}]^W$ acts on each v_i by 0 and sends w_{-2k} to $2v_{-2k}$. Then x acts by moving the diagram to the left, i.e. $x(v_k) = 0$ and $x(w_k) = v_k$. $\tilde{T}_{0\to -1}(P(-2)) \simeq \Delta(-1)^2$ as $U(\mathfrak{sl}_2)$-module. The action $T_{0\to -1}(x)$ where we consider x as endomorphism of $P(-2)$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. After twisting with ρ_{-1} we get a matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ that corresponds to the $*$-action of x. In particular we get $\operatorname{End}(P(-2)) \simeq \operatorname{End}(\tilde{T}_{0\to -1}(P(-2))) \simeq \mathbb{C}[x]/(x^2)$ *and the first isomorphism is induced by $\tilde{T}_{0\to -1}$. This is an easy case of Theorem 2.2.*

Proposition 3.15. *The two actions of $\mathbb{C}[\mathfrak{h}]^W$ on $N = T_{\lambda\to\mu}\operatorname{Res}_\lambda(M)$ (one coming from the shifted $\mathbb{C}[\mathfrak{h}]$-action and one coming from the central inclusion $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow U(\mathfrak{g})$) coincide.*

By Proposition 3.10, an equivalent formulation of this proposition is that the actions of $\mathbb{C}[\mathfrak{h}]^{W_\mu} \subset \mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}]^{W_\mu} \subset \tilde{U}^{W_\mu}$ coincide.

Proof. The proof is in several steps. Analogously to the proof of Theorem 4.7 in Chris's notes we may assume that $\lambda - \mu$ is dominant.

Step 1. The category $\tilde{\mathcal{O}}_\lambda$ has enough projectives, the indecomposable ones are $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}} P(\lambda')$ for $\lambda' \in W \cdot \lambda$. It is enough to prove the statement for a projective M since any object in $\tilde{\mathcal{O}}_\lambda$ is covered by a projective. We will consider the projectives of the form $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}} \operatorname{pr}_\lambda(V \otimes \Delta(\lambda)) \in \tilde{\mathcal{O}}_\lambda$.

For these objects M the proof is by a deformation argument – we reduce the proof to the case when relevant infinitesimal blocks of $\mathcal{O}$ are semisimple by deforming the parameter λ.

Step 2. Pick a very small positive number ϵ and consider $z \in \mathbb{C}$ with $|z| < \epsilon$. Consider $\lambda_z := \lambda + z(\lambda + \rho)$. For $z \neq 0$, we have $W_{\lambda_z} = W_\lambda$ and different elements in $W \cdot \lambda_z$ are non-comparable with respect to the standard order $\leqslant$. In particular, the infinitesimal block $\mathcal{O}_{\lambda_z}$ is semisimple with $|W/W_\lambda|$ objects.

Step 3. Let us set a new notation $\overline{\operatorname{pr}}_{\lambda_z}(V) = \bigoplus_{\nu_i} \operatorname{pr}_{\lambda_z + \nu_i}(V)$ where the sum is taken over all ν_i such that $\lambda + \nu_i \in W \cdot \lambda$. In other words, $\overline{\operatorname{pr}}_{\lambda_z}$ projects to infinitesimal blocks corresponding to the central characters of $\lambda_z + \nu$ (where ν is a weight of V) that are close to the central character of λ. Now observe that $\overline{\operatorname{pr}}_{\lambda_z}(V \otimes \Delta(\lambda_z))$ is a flat deformation of $\operatorname{pr}_\lambda(V \otimes \Delta(\lambda))$ (the Verma subquotients that survive in $\overline{\operatorname{pr}}_{\lambda_z}(V \otimes \Delta(\lambda_z))$ and in $\operatorname{pr}_\lambda(V \otimes \Delta(\lambda))$ are labeled by the same weights).

Step 4. Set $\mu_z = \lambda_z + \mu - \lambda$. Let $\overline{\operatorname{pr}}_{\mu_z} = \bigoplus_{\nu_i} \operatorname{pr}_{\lambda_z + \nu_i}(V)$ where ν_i are as in Step 3. Again, we note that

$$\overline{\mathrm{pr}}_{\mu_z}(L(\lambda-\mu)^*\otimes\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}}\overline{\mathrm{pr}}_{\lambda_z}(V\otimes\Delta(\lambda_z))) \tag{3.1}$$

is a flat deformation of

$$T_{\lambda\to\mu}[\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}}\mathrm{pr}_\lambda(V\otimes\Delta(\lambda))]=\mathrm{pr}_\mu[L(\lambda-\mu)^*\otimes\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}}\mathrm{pr}_\lambda(V\otimes\Delta(\lambda))]. \tag{3.2}$$

This is for the same reason as in Step 3. It follows that it is enough prove the coincidence of the two actions of $\mathbb{C}[\mathfrak{h}]^W$ on the deformed module 3.1 (for $z\neq 0$) (then we will be done by continuity).

Step 5. The point of this reduction is that $\overline{\mathrm{pr}}_{\lambda_z}(V\otimes\Delta(\lambda_z))$ splits into the sum of Vermas (=simples). Pick $w\in W$. It is enough to prove the coincidence of the actions on

$$\overline{\mathrm{pr}}_{\mu_z}(L(\lambda-\mu)^*\otimes\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}}\Delta(w\cdot\lambda_z)). \tag{3.3}$$

As in Step 3, this object is $\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}}\Delta(w\cdot\mu_z)$. So $\mathbb{C}[\mathfrak{h}]^W\subset U(\mathfrak{g})$ acts on (3.3) via μ_z. On the other hand, $\mathbb{C}[\mathfrak{h}]^W\subset\mathbb{C}[\mathfrak{h}]$ acts on $\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{W_\lambda}}\Delta(w\cdot\lambda_z)$ via λ_z and hence it acts on (3.3) by μ_z as well. ☐

From the construction we get that the following diagram is commutative.

$$\begin{tikzcd}
\tilde{\mathcal{O}}_\lambda \arrow[r, "\tilde{T}_{\lambda\to\mu}"] \arrow[d, "Res_\lambda"] & \tilde{\mathcal{O}}_\mu \arrow[d, "Res_\mu"] \\
\mathcal{O}_\lambda \arrow[r, "T_{\lambda\to\mu}"] & \mathcal{O}_\mu
\end{tikzcd}$$

.

Extended translation functors are transitive in the following sense: $\tilde{T}_{\lambda\to\nu}=\tilde{T}_{\mu\to\nu}\circ\tilde{T}_{\lambda\to\mu}$.

Remark 3.16. *We have a generalization of Example 3.14 to the case when μ is on the single wall $\ker\alpha_i^\vee$. We set $\tilde{\Delta}_{w,i}=T_{\mu\to 0}\Delta(w\cdot\mu)$. Suppose that $l(ws_i)<l(w)$, so we have an exact sequence $0\to\Delta(ws_i\cdot 0)\to\tilde{\Delta}_{w,i}\to\Delta(w\cdot 0)\to 0$. Analogously to Example 3.14 the functor $\tilde{T}_{0\to\mu}$ gives an isomorphism $\mathrm{End}(\tilde{\Delta}_{w,i})\simeq\mathrm{End}(\tilde{T}_{0\to\mu}(\tilde{\Delta}_{w,i}))$ where $\tilde{T}_{0\to\mu}(\tilde{\Delta}_{w,i})=\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}}\Delta(w\cdot\mu)$, so the endomorphsim algebra is $\mathbb{C}[x]/(x^2)$. In particular, we have that the root $\alpha_i\in\mathbb{C}[\mathfrak{h}]_+$ acts nontrivially on $\tilde{\Delta}_{w,i}$. This action kills the bottom Verma $\Delta(ws_i\cdot 0)$ and sends $\Delta(w\cdot 0)$ to $\Delta(ws_i\cdot 0)$ by the unique non-trivial homomorphism.*

3.6. Properties of $\mathcal{F}$ and its adjoints.

Let us write $\mathcal{F}$ for $\tilde{T}_{0\to-\rho}$. This is a functor $\mathcal{O}_0\to\tilde{\mathcal{O}}_{-\rho}\simeq C$-mod. As we have pointed out already, it admits a left adjoint $\mathcal{F}^!$ (if A,B are finite dimensional algebras, then any exact functor $\mathcal{F}:A\text{-mod}\to B\text{-mod}$ has the form $\mathrm{Hom}_A(P,\bullet)$, where P is a projective A-module with a homomorphism $B\to\mathrm{End}_A(P)^{opp}$, then the left adjoint is $P\otimes_B\bullet$). By a dual argument, $\mathcal{F}$ also admits a right adjoint, $\mathcal{F}^*$.

Lemma 3.17. *The following are true:*

1. $\mathcal{F}(L(w\cdot 0))=0$ *if* $w\neq w_0$ *(the longest element) and is the unique simple* C*-module* $\mathbb{C}$*, else.*
2. $\mathcal{F}(\Delta(w\cdot 0))=\mathbb{C}$ *for all* $w\in W$*.*
3. $\mathcal{F}^!(C)=P_{min}$*.*
4. $\mathcal{F}^*(C)=P_{min}$*.*

Proof. By the construction, $\mathsf{frg}\circ\mathcal{F}=T$, so (1) and (2) follow from the properties of T from Chris's talk.

To prove (3) we note that $T^*=T^!=\mathcal{F}^!\circ\mathrm{Res}^!$. We have $\mathrm{Res}^!(\mathbb{C})=C$ because C is projective cover of $\mathbb{C}$. Indeed, $\mathrm{Hom}_{C\text{-mod}}(C,X)=\mathrm{Hom}_{\mathrm{Vect}}(\mathbb{C},\mathrm{Res}\,X)$. Therefore $\mathcal{F}^!(C)=T^*(\mathbb{C})=P_{min}$ by Proposition 3.4.

Let us prove (4). By (5) of Lemma 3.9, C is an injective C-module. Therefore C is injective envelope of $\mathbb{C}$ and $\mathrm{Res}^*(\mathbb{C})=C$. Analogously $\mathcal{F}^*(C)=T^*(\mathbb{C})=P_{min}$. ☐

From (4) we get a natural map $\phi : C \simeq \mathrm{Hom}_{C\text{-mod}}(C, C) \to \mathrm{Hom}_{\mathcal{O}}(P_{min}, P_{min})$.

Proposition 3.18. *We have* $\mathcal{F}^*(\mathbb{C}) = \Delta(0)$ *(and, similarly,* $\mathcal{F}^!(\mathbb{C}) = \nabla(0)$*).*

Proof. The proof is in several steps.

Step 1. We can consider $\alpha_1, \dots, \alpha_k \in \mathfrak{h}^*$ as elements of C. Let $\psi_i = \phi(\alpha_i)$ be the corresponding endomorphism of P_{min}. We have an embedding $\mathbb{C} \to C$ as the socle, i.e. the intersection of kernels of all α_i because they generate the maximal ideal of C. $\mathcal{F}^*$ is left exact functor, so $\mathcal{F}^*(\mathbb{C})$ is the intersection of kernels of all ψ_i. We need to show that this intersection coincides with $\Delta(0)$. Note that $\Delta(0)$ is in the kernel of any ψ_i. Indeed, order the labels $w_1, \dots, w_N$ in W so that $w_i \preceq w_j \Rightarrow i \geq j$. Then we have a canonical standard filtration $P_{min} = P^0 \supset P^1 \dots \supset P^N = \{0\}$ with $P^{i-1}/P^i = \Delta(w_i \cdot 0)$. This filtration is preserved by every endomorphism (there are no Hom's from lower to higher Vermas). In particular, all ψ_i's preserve the filtration. Since each of them is nilpotent, they kill $\Delta(0)$.

Step 2. Note that P_{min} is filtered with successive quotients $\tilde{\Delta}_{w,i}$, for $w \in W/\{1, s_i\}$. This is because $P_{min} = T^*\Delta(-\rho) = T_{\mu\to 0}(T_{-\rho\to\mu}\Delta(-\rho))$, $\tilde{\Delta}_{w,i} = T_{\mu\to 0}\Delta(w \cdot \mu)$. For reasons similar to Step 1, each of the filtration terms is preserved by ψ_i. We claim that on each of the direct summand $\tilde{\Delta}_{w,i}$ of the associated graded of the filtration ψ_i is nonzero.

Step 3. Let μ be on the wall corresponding to the root α_i. From the transitivity $\tilde{T}_{0\to-\rho} = \tilde{T}_{\mu\to-\rho}\tilde{T}_{0\to\mu}$. Therefore the map $C \to \mathrm{End}(P_{min})$ factors through $C \to \mathrm{End}(\tilde{T}^*_{\mu\to-\rho}(C)) \to \mathrm{End}(P_{min})$. Analogously to (4) of the previous lemma $\tilde{T}^*_{\mu\to-\rho}(C) = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}} P_{min,\mu}$. This object is filtered by $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}} \Delta(w \cdot \mu)$. Note that the action of α_i on the latter is induced from the multiplication on α_i. We have $\tilde{T}_{0\to\mu}\tilde{\Delta}_{w,i} = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}} \Delta(w \cdot \mu)$ (see Remark 3.16). The induced homomorphism $\mathrm{End}(\tilde{\Delta}_{w,i}) \to \mathrm{End}(\tilde{T}_{0\to\mu}\tilde{\Delta}_{w,i})$ is an isomorphism. The endomorphism $\underline{\psi}_i$ of $T^*_{\mu\to-\rho}(C)$ preserves the filtration by $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^s} \Delta$'s and is nonzero on each of the factors. Therefore the endomorphism $\psi_i = T^*_{0\to\mu}(\underline{\psi_i})$ of P_{min} is nonzero on each $\tilde{\Delta}_{w,i}$.

Step 4. Now we are ready to prove the claim of Step 1. Let K stand for the intersection of the kernels of the ψ_i's. Pick minimal j such that $K \not\subset P^{j+1}$ for a filtration $P = P^0 \supset P^1 \supset \dots \supset P_N = \{0\}$ as in Step 1. We can assume that j is minimal for all such filtrations. That means that if $i < j$ then $w_j \prec w_i$. Suppose that $w_j \neq id$. Then there is i such that $w_{j'} := w_j s_i \prec w_j$. Note that if $\Delta(u \cdot 0)$ occurs in P^j then $\Delta(us_i \cdot 0)$ does. Indeed, otherwise $w_j \prec us_i$. As $w_j s_j \prec w_j$ that implies $w_j \prec u$ and we get a contradiction. Therefore P^j is filtered by $\tilde{\Delta}_{u,i}$ where $w \not\prec u$ and $w \not\prec us_i$ and $\tilde{\Delta}_{w_j,i}$ is the top factor. Consider the projection of K on $\tilde{\Delta}_{w_j,i}$. It has non-trivial projection to the Verma quotient $\Delta(w \cdot 0)$, so by Step 3 is not annihilated by ψ_i. The contradiction finishes the proof. $\square$

3.7. **Completion of the proof.** First, we claim that $\mathcal{F}(P_{min}) = C$. By Proposition 3.8, the standard filtration of P_{min} contains $|W|$ Vermas and by (2) of Lemma 3.17, the image of each Verma under $\mathcal{F}$ is one-dimensional. So $\dim \mathcal{F}(P_{min}) = |W| = \dim C$.

Now $\dim \mathrm{Hom}_C(\mathcal{F}(P_{min}), \mathbb{C}) = \dim \mathrm{Hom}_{\mathcal{O}}(P_{min}, \mathcal{F}^*\mathbb{C}) = \dim \mathrm{Hom}_{\mathcal{O}}(P_{min}, \Delta(0)) = 1$ by Corollary 3.6. Since C is projective, the homomorphism $C \to \mathbb{C}$ lifts to $C \to \mathcal{F}(P_{min})$. Since the homomorphism $\mathcal{F}(P_{min}) \to \mathbb{C}$ is unique up to proportionality, we see that $C \to \mathcal{F}(P_{min})$ is an epimorphism. Since the dimensions coincide, $\mathcal{F}(P_{min}) = C$.

Now consider the natural homomorphism $P_{min} \to \mathcal{F}^* \circ \mathcal{F}(P_{min}) = P_{min}$. Applying $\mathcal{F}$ we get a surjective homomorphism. Since $\mathcal{F}$ does not kill the head of P_{min}, we conclude that $P_{min} \twoheadrightarrow \mathcal{F}^* \circ \mathcal{F}(P_{min}) = P_{min}$. But any surjective endomorphism of P_{min} is an isomorphism.

Once $P_{min} \xrightarrow{\sim} \mathcal{F}^* \circ \mathcal{F}(P_{min})$, we see that $\mathrm{End}_{\mathcal{O}}(P_{min}) \xrightarrow{\sim} \mathrm{End}_C(\mathcal{F}(P_{min})) = \mathrm{End}_C(C) = C$.

As a conclusion we get that the Soergel functor $\mathbb{V} : \mathcal{O}_0 \to \text{mod-}\mathrm{End}_{\mathcal{O}}(P_{min})$ is, in fact, the extended translation functor $\tilde{T}_{0\to-\rho} : \mathcal{O}_0 \to C\text{-mod}$.

For the subsequent applications (to prove that $\mathbb{V}$ is fully faithful on the projective objects) let us point out that we have seen above that the natural homomorphism $P_{min} \to \mathbb{V}^* \circ \mathbb{V}(P_{min})$ is an isomorphism.

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