AHAHA: Preliminary results on p-adic groups and their representations.

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1 Introduction and motivation

Let k be a locally compact non-discrete field with non-Archimedean valuation (Say, just the p-adic numbers $\mathbb{Q}_p$), $\mathcal{O}$ its ring of integers (say $\mathbb{Z}_p$) and $\mathcal{P}$ a generator for the maximal ideal of $\mathcal{O}$ (i.e. p). In future lectures we will see how Hecke algebras relate to the representation theory of reductive algebraic groups over these fields.

The main example to keep in mind is $G = GL(n,k)$. When I talk about open compact subgroups you should think of $GL(n,\mathcal{O})$ and subgroups thereof. Other important subgroups to keep in mind are parabolic subgroups and unipotent subgroups.

This will be a basic introduction to the theory of these groups and their representation theory, in particular I will focus on aspects of the theory that are different from say finite groups or Lie groups. Things will be stated in some generality, but I will try to say what the cases we actually care about are.

2 Preliminaries on l-spaces and l-groups

An *l-space* is a topological space that is Hausdorff, locally compact, and zero dimensional. By zero dimensional we mean that each point has a fundamental system of open compact neighborhoods.

An *l-group* is a topological group such that the identity element has a fundamental system of neighborhoods consisting of open compact subgroups. It can be checked that a topological group is an l-group iff it is an l-space.

Remark: A compact l-group is the same thing as a pro-finite group. Hence l-groups are also sometimes known as locally pro-finite groups.

Remark: For some results we may also assume that G is countable at infinity, that is G is a countable union of open compact sets. This is equivalent to G/N being countable for some open compact N.

Motivating example: In our motivating situation k be a locally compact non-discrete field with non-Archimedean valuation, $\mathcal{O}$ its ring of integers and $\mathcal{P}$ a generator for the maximal ideal of $\mathcal{O}$. Then $G = GL(n,k)$ is an l-group with a fundamental system of open compact neighborhoods $N_i = 1 + \mathcal{P}^i \cdot M(n,\mathcal{O})$.

Some basic topological facts:

- If X is an l-space and $Y \subset X$ is a locally closed subset of X then Y is an l-space in the induced topology.
- If K is a compact subset of an l-space X then any covering of K by open sets has a finite refinement by pairwise disjoint open compact subsets. (This is sometimes a definition of zero topological dimension)
- If G is an l-group and H is a closed subgroup of G, then G/H (or $H\backslash G$) is an l-space under the quotient topology.

Let X be an l-space. We will let $S(X)$ be the set of locally constant complex valued functions with compact support on X, these are called *Schwartz functions*. The dual space $S^*(X)$ is the space of *distributions*. Given an open subset $Y \subset X$ we have the following exact sequence:

$$0 \to S(Y) \to S(X) \to S(X\backslash Y) \to 0$$

Where the first map is extension by zero and the second is restriction. Surjectivity follow from the zero dimensional requirement. We get an analogous exact sequence for distributions by taking duals.

We say that a distribution is *finite* if it has compact support, and denote the set of finite distributions by $S_c^*(X)$. Given a sheaf $\mathcal{F}$ of vector spaces on an l-space X we can look at its compactly supported sections and dualize to talk about distributions on $\mathcal{F}$.

2.1 Haar measure stuff

Now let G be an l-group and consider the action of G on itself by left translations. Then there exists a unique (up to scalars) G-invariant distribution $\mu_G \in S^*(X)$. More explicitly this means:

$$\int_G f(g_0 g) d\mu_G(g) = \int_G f(g) d\mu_G(g) \quad \forall f \in S(X),\ g_0 \in G$$

We also require that this be positive, that is the integral of any positive real function is positive. This is called the (left) *Haar Measure*. The right Haar measure is defined similarly.

Since the right and left actions of G on itself commute it follows that right multiplication by $g \in G$ sends μ_G to some multiple $\Delta_G(g)$. This function $\Delta_G(g)$ is easily seen to be a character, called the modulus. In particular we get that $\Delta_G(g)^{-1}\mu_G$ is a right Haar measure (for the opposite group).

Any compact subgroup gets mapped to a compact subgroup of $R^*_{>0}$ and hence gets mapped to 1. If G is generated by compact subgroups then $\Delta_G(g) = 1$ and G is said to be *unimodular* (This is the case for $GL(n, k)$).

Warning: In general it is not always possible to define a G invariant Haar measure on G/H. This is possible iff $\Delta = \Delta_G/\Delta_H = 1$. This can be fixed to some extent by defining "$\mathcal{O}(G/H)$" to be functions that transform by Δ under H instead of being H-invariant, this space will then have a Haar measure.

2.2 The convolution algebra

Given two distributions $T_1, T_2 \in S_c^*(G)$ define their convolution $T_1 * T_2$ by

$$\int_G f(g) d(T_1 * T_2)(g) = \int_{G\times G} f(g_1 g_2) d(T_1 \otimes T_2)(g_1, g_2)$$

Where we view $T_1 \otimes T_2$ as a distribution on $G \times G$ in the natural way. It's easy to see that the support of $T_1 * T_2$ is contained in $supp(T_1) \cdot supp(T_2)$, so in particular $T_1 * T_2 \in S_c^*(G)$.

This turns $S_c^*(G)$ into an associative algebra with unit element δ_e the Dirac delta distribution at the group identity element. The mapping $g \to \delta_g$ gives us an inclusion of G into $S_c^*(G)$, if G were finite then this gives an isomorphism between $\mathbb{C}(G)$ and $S_c^*(G)$.

Moreover the map $f \to f\mu_G$ lets us identify $S(G)$ with the space $\mathcal{H}(G)$ of finite distributions that are locally constant on the left, that is distributions T such that there exists an open subgroup $N \in G$ such that $NT = T$. $\mathcal{H}(G)$ is a two sided ideal of $S_c^*(G)$.

This lets us carry over the convolution operation to $S(G)$ explicitly this operation is given by the familiar formula:

$$f_1 * f_2(g_0) = \int_G f_1(g) f_2(g^{-1} g_0) d\mu_G(g)$$

Remark: We will stick with the notation $\mathcal{H}(G)$ when referring to $S(G)$ as a (non-unital) algebra with this new multiplication. This is sometimes called the Hecke algebra of G.

3 Representations of l-groups

We will be talking about complex representations of l-groups. For notation we will write $\pi = (\pi, G, V)$ for "π is a representation of the l-group G on a complex vector space V". I may further just shorten it to V. We will put no topology on our vector spaces, and have the usual notion of irreducible.

Some definitions:

A representation (π, G, V) is *algebraic* (or *smooth*) if for any $v \in V$ we have that $stab(v)$ is open in G. For an arbitrary representation (π, G, V) the set of such vectors forms a subrepresentation called the *algebraic part* of π.

We say a representation is *admissible* if it is algebraic and for any open subgroup $N \in G$ the space V^N of N invariant vectors is finite dimensional.

Example For any l-group G acting on an l-space X the action of G on $S(X)$ is algebraic but the action of G on $C^\infty(X)$ is not necessarily algebraic. (Its algebraic part consists of those functions that are locally constant on the left)

Let (π, G, V) be an algebraic representation and $T \in S_c^*(G)$ be a finite distribution on G. We define an operator $\pi(T)$ on V thinking of $g \to \pi(g)v$ as a V valued function on G and integrating with respect to T. That is, we define:

$$\pi(T)v = \int_G \pi(g)v dT(g)$$

This upgrades V to being a representation of $S_c^*(G)$, and the copy of G we have sitting inside $S_c^*(G)$ acts the right way, that is, $\pi(\delta_g) = \pi(g)$.

If $H \subset G$ is a compact subgroup then define δ_H to be the Haar measure on H (normalized so that H has volume 1) considered as a measure on G. In particular $\pi(\delta_H)$ is just the projection onto V^H, the space of H invariant vectors. To see this just note that $\pi(\delta_H)$ acts by 0 on the subspace spanned by all vectors of the form $\pi(h)v - v$ for $h \in H$, and that H acts trivially on the quotient by this subspace.

Let (π, G, V) be an algebraic representation of G. Then can also think of it a representation of $\mathcal{H}(G)$ with the property that $V = \cup \pi(\delta_N)V \quad (= \cup V^N)$ where the union is over all compact open subgroups $N \in G$. In fact we have a converse to this and we get that the category of algebraic representation is equivalent to the subcategory of $\mathcal{H}(G)$ modules such that $V = \cup\pi(\delta_N)V$

Proof: Just identify G with the delta functions in $S_c^*(G)$ and define an action of $T \in S_c^*(G)$ by $\pi(T)v = \pi(T * \delta_N)$ for some open compact N such that $v \in V^N$.

3.1 The representations π_N

Let $N \subset G$ be an open compact subgroup of an l-group G. Let $\mathcal{H}_N = \delta_N * S_c^*(G) * \delta_N$ be the set of N bi-invariant compactly supported distributions, this is a unital algebra with identity element δ_N. For an algebraic representation (π, G, V) let π_N be the representation of $\mathcal{H}_N$ on V^N.

Proposition: π is irreducible iff for all open compact subgroups N: $\pi_N = 0$ or π_N is irreducible.

Proof: Suppose π is irreducible. Take any two vectors $v_1, v_2 \in V^N$, since V is irreducible there exists $T \in \mathcal{H}(G)$ such that $\pi(T)v_1 = v_2$ but then it follows that $\pi(\delta_N * T * \delta_N)v_1 = v_2$ so V^N is irreducible. Conversely, if $V' \subset V$ is a sub representation then for N small enough we can find N invariant vectors both inside and outside V', so V'^N is a sub representation of V^N.

Proposition: Given two irreducible representations π_1, π_2 such that $\pi_{1N} \simeq \pi_{2N} \not\simeq 0$ then we have $\pi_1 \simeq \pi_2$ as G representations.

Proof: Suppose $j : V_1^N \to V_2^N$ is an isomorphism of $\mathcal{H}_N$ modules. Then consider $W := \{(v, jv)\} \subset V_1^N \oplus V_2^N$ and consider the G module $\tilde{W}$ generated by W in $V_1 \oplus V_2$. We have that $\tilde{W}^N = W$ so in particular $\tilde{W}$ is neither contained in nor contains V_1 or V_2. Since these are irreducible the projections must be isomorphisms so $V_1 \simeq \tilde{W} \simeq V_2$.

Proposition: Given any irreducible representation (τ, W) of $\mathcal{H}_N$ there exists an irreducible algebraic representation of G such that $\tau \simeq \pi_N$.

Proof: W is irreducible so $W \simeq \mathcal{H}_N/I$ for some left ideal I. Now consider $\mathcal{H}(G)$ and a left module over itself and let V_1, V_2 be the submodules generated by $\mathcal{H}_N$ and I. Then $V_1^N \simeq \mathcal{H}_N$ and $V_2^N \simeq I$. So if we let $V_3 \simeq V_1/V_2$ we get $V_3^N \simeq W$. Taking an appropriate irreducible factor of this gives us what we want.

3.2 Contragradient (i.e. dual) representations

Let (π, G, V) be an algebraic representation. As usual we get dual representation (π^*, G, V^*) defined by $(\pi^*(g)\xi)(v) = \xi(\pi^{-1}(v))$ on the full dual. Taking the algebraic part of this we get a representation $(\tilde{\pi}, G, \tilde{V})$ called the contragradient representation, or just the algebraic dual.

For any compact subgroup $H \subset G$ and $\xi \in (V^*)^H$ we have that $\xi(v) = \xi(\pi(\delta_H)v)$ for all $v \in V$. In particular this implies that $(V^*)^H \simeq (V^H)^*$. Moreover if H is an open compact group then $\tilde{V}^H \simeq (V^*)^H \simeq (V^H)^*$ (as any H invariant linear form will necessarily be algebraic since H is contained in its stabilizer).

For $T \in S_c^*(G)$ define $\check{T}$ to be the distribution obtained from T by means of the map $g \mapsto g^{-1}$. For all $\xi \in \tilde{V}$ and $v \in V$ we have that $\langle \tilde{\pi}(T)\xi, v\rangle = \langle \xi, \pi(\check{T})v\rangle$. Indeed:

$$\langle \tilde{\pi}(T)\xi, v\rangle = \int_G \langle \tilde{\pi}(g)\xi, v\rangle dT(g) = \int_G \langle \xi, \tilde{\pi}(g^{-1})v\rangle dT(g) = \int_G \langle \xi, \tilde{\pi}(g)v\rangle d\check{T}(g)$$

Proposition Now lets assume that π is admissible. We have that:

1. $\tilde{\pi}$ is also admissible.
2. The natural map $V \to \tilde{\tilde{V}}$ is an isomorphism.
3. π is irreducible iff $\tilde{\pi}$ is.

Proof: (1) and (2) follow from the fact that $(\tilde{V})^N = (V^N)^*$ and these are all finite dimensional. For (3) If $W \in V$ were a submodule then those linear functionals vanishing on W form a subrepresentation of $\tilde{V}$.

3.3 Characters

Let (π, G, V) be an admissible representation. Then for any $T \in \mathcal{H}(G)$ we have that $\pi(T)$ has finite rank. (To see this just note that T is a finite linear combination of G translates of the distributions δ_N which are just projections onto the finite dimensional spaces V^N). In particular it makes sense to talk about the trace $tr\ \pi(T)$. Hence we can define the trace distribution $tr\pi$ by fixing a haar measure μ_G and letting

$$(tr\ \pi)(f) = tr\ \pi(f\mu_G)$$

This is called the character of π

Proposition: Given $\pi_1, \pi_2, \ldots, \pi_k$ pairwise inequivalent irreducible admissible representations of an l-group then their characters $tr\ \pi_1, tr\ \pi_2, \ldots, tr\ \pi_k$ are linearly independent. In particular two irreducible admissible representations are the same iff they have the same character.

Proof (sketch) Choose an open compact subgroup N such that the representations π_i^N are all nonzero. They are all irreducible and pairwise inequivalent finite dimensional representations of $\mathcal{H}_N$, and the result for finite dimensional representations is standard.

3.4 Induced representations and Frobenius reciprocity

Let $H \subset G$ be a closed subgroup of an l-group G, and let (ρ, H, V) be an algebraic representation. We will let $L(G, \rho)$ be the space of functions $f : G \to V$ such that:

1. $f(hg) = \rho(h)f(g)$ for all $h \in H$ and $g \in G$
2. There exists an open compact subgroup $f(gn) = f(g)$ for all $g \in G$ and $n \in N$.

And call this the induced representation $Ind_H^G(\rho)$, condition (2) is there to ensure this is algebraic.

Inside $L(G,\rho)$ we have a subspace $S(G,\rho)$ of those functions f that are *finite* modulo H, that is, there exists a compact set $K_f \in G$ such that $supp(f) \in H \cdot K_f$. We call this the *finitely induced* representation $ind_H^G(\rho)$.

Remark These two constructions are related by the identity $\widetilde{ind_H^G(\rho)} \simeq Ind_H^G(\Delta_G/\Delta_H\tilde{\rho})$. The pairing is given by integrating $\tilde{f}(g)f(g)$ over $H\backslash G$ with the twisted Haar measure I mentioned earlier. In the case that $\Delta_G/\Delta_H = 1$ everything works out much nicer.

In the case where G is compact modulo H (for example if H is a parabolic subgroup) then these two versions of induction coincide and send admissible representations to admissible representations. To see this just note that for any open compact N we have that the set of double cosets $N\backslash G/H$ is finite.

We get the the usual properties of induction functors that we would expect:

1. Ind_H^G and ind_H^G are exact functors.
2. $Ind_H^G \circ Ind_F^H = Ind_F^G$ and similarly for *ind*
3. We now have two versions of Frobenius reciprocity:

$$Hom_G(\pi, Ind_H^G(\rho)) \cong Hom_H(\pi|_H, \rho)$$

$$Hom_G(ind_H^G(\rho), \tilde{\pi}) \cong Hom_H(\Delta_H/\Delta_G\rho, \widetilde{(\pi|_H)})$$

Remark: Of particular importance is when H is a parabolic subgroup with Levi decomposition $H = MN$, and we are inducing up modules on which N acts trivially. This is called parabolic induction.

3.5 Functors $V_{G,\theta}$

Remark: For this section we mostly want to think of H as being the unipotent radical of some parabolic subgroup. This will give us so called parabolic restriction.

Let H be an l-group and θ be a character of H (i.e. a one dimensional algebraic representation). Then for any representation (π, H, V) define $V(H,\theta)$ to be the subspace spanned by all vectors of the form $\pi(h)v - \theta(h)v$. Then let $V_{H,\theta} := V/V(H,\theta)$, if θ is the trivial character we will just write this as V_H the space of coinvariants.

- This is a θ twisted version of coinvariants so as we expect, the dual notion is a twisted version of invariants. That is: $V^*_{H,\theta} = \{\xi \in V^* | \pi^*(h)v = \theta^{-1}(h)v\}$.

- If we twist the representation by θ^{-1}, then we can often reduce to the case where θ is trivial. That is: $(V_\pi \otimes \mathbb{C}_{\theta^{-1}})_H \simeq (V_\pi)_{H,\theta}$

- Suppose G is a closed subgroup of some l-group G. Define a θ -normalizer: $\mathrm{Norm}_G(H,\theta) = \{g \in G | ghg^{-1} \in H \text{ and } \theta(ghg^{-1}) = \theta(h)\ \forall h \in H\}$. For any representation V of G, $\mathrm{Norm}_G(H,\theta)$ preserves $V(H,\theta)$ and hence acts on $V_{H,\theta}$. (If H is the unipotent radical of a parabolic P then $Norm_G(H,1) = P$)

Now assume that H is exhausted by its compact subgroups, that is, every compact subset of H is contained in some compact subgroup of H. In this case we have a convenient way to describe $V(H,\theta)$:

Theorem: (Jacquet and Langlands) A vector $v \in V$ lies in $V(H,\theta)$ if and only if there exists a compact subgroup $N \in H$ such that:

$$\pi(\theta^{-1} \cdot \delta_N)v = \int_N \theta^{-1}(h)\pi(h)v d\mu_N(h) = 0$$

Note that the case when $\theta = 1$ we already looked at the kernel of $\pi(\delta_N)$, the general case reduces to this by twisting by θ.

As a corollary of this, under the same conditions we see that if $V' \subset V$ is a sub representation then $V'(H,\theta) = V' \cap V(H,\theta)$. This then implies that $V \mapsto V_{H,\theta}$ is an exact functor. (Without the assumption it may just be right exact)